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Variational Methods for Evolution

Organised by
Alexander Mielke, Berlin
Felix Otto, Leipzig
Giuseppe Savaré, Pavia
Ulisse Stefanelli, Pavia

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Abstract. The meeting focused on the last advances in the applications of variational methods to evolution problems governed by partial differential equations. The talks covered a broad range of topics, including large deviation and variational principles, rate-independent evolutions and gradient flows, heat flows in metric-measure spaces, propagation of fracture, applications of optimal transport and entropy-entropy dissipation methods, phase-transitions, viscous approximation, and singular-perturbation problems.

Mathematics Subject Classification (2000): 30Lxx, 35A15, 35Qxx, 49xx, 74xx, 76xx.

Introduction by the Organisers

It is well known that the study of many important evolution problems gains benefit by adopting a variational point of view. Variational methods can be quite helpful to better understand the intimate structure of the problem, to derive new properties concerning existence, uniqueness, stability, approximation, and long-time behaviour, to guess new estimates, to find the right or more appropriate formulations, to produce new kind of solutions, to explain the relationships between different approaches or techniques, to derive or justify some equations in terms of general principles and stochastic models, to study the stability of a system in terms of a set of parameters or to capture its behavior in a singular perturbation limit.

The workshop, organized by Alexander Mielke (WIAS, Berlin), Felix Otto (Max Planck Institute, Leipzig), Giuseppe Savaré (Univ. Pavia) and Ulisse Stefanelli

(IMATI-CNR, Pavia), aimed to present many new, striking and promising achievements in this wide area, thanks to the contribution of over 50 participants with broad geographic representation and a variety of research fields, each revealing different methodology, interests, and level of abstraction.

One of the organizers' main goal was to generate a strong interaction between various subjects and people with different backgrounds, covering calculus of variations, optimal transport, phase transitions, fluid mechanics, materials science, stochastic calculus, and models. Approximately 25 talks of varying lengths were delivered by experts in the fields but also by quite a number of young post-docs and PhD students and stimulated a lot of discussions in a friendly and inspiring atmosphere, that contributed to the overall success of the meeting.

Among the main themes presented during the workshop, we quote here

- the interplay between the theory of large deviation in stochastic analysis on the one hand and variational principles on the other; e. g. an interpretation of the Wasserstein gradient flow structure in terms of large deviation, rigorous low-temperature bounds for the relaxation in multi-well energy landscapes, singular limits of rate functionals;
- the energetic approach to rate-independent evolution problems and various viscous regularizations, with applications to damage, fracture, and phase transitions;
- crack propagation, e. g. an emerging analysis of fully dynamic models, novel functional analytic aspects of the quasi-stationary models, and the analysis of alternative propagation criteria;
- the classic or new gradient-flow structure of various linear and nonlinear PDE's;
- the application of optimal transport techniques to solve, study, or approximate efficiently evolution equations from fluid mechanics, particle interaction, quantum problems;
- entropy-entropy dissipation methods for evolution and their link with sharp functional inequalities;
- the interplay between evolution problems and geometry in metric-measure spaces; e. g. a clear distinction between Riemannian and Finsler settings is emerging;
- phase-transitions, viscous approximation, and singular-perturbation problems.

Special event

On Thursday afternoon, the Oberwolfach Prize was awarded to *Nicola Gigli and László Székelyhidi* for their excellent achievements in Analysis and Applied Mathematics. The award presentation by Gert-Martin Greuel, Director of the Institute, and by Manfred Feilmeier of the Oberwolfach Foundation, was followed by the Laudationes of the two price-winners. Luigi Ambrosio explained the crucial contributions of Gigli concerning the metric theory of gradient flows, the structure of the Wasserstein space, and the heat flow in non-smooth metric-measure spaces.

Stefan Müller explained the fundamental new insights of Székelyhidi into the theory of nonlinear partial differential equations and their applications in continuum mechanics, focusing in particular on compensated compactness, partial regularity and its failure.

Nicola Gigli then gave a lecture on his more recent results on metric measure spaces with a lower Ricci curvature bound and László Székelyhidi presented his recent achievements on the Euler equations, in particular the Onsager conjecture, and its relation to the Nash-Kuiper embedding in geometry. In the evening, there was a festive dinner.

Workshop: Variational Methods for Evolution

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Abstracts

A diffusion equation for 3D divergence-free vector fields

Yann Brenier

Introduction. We show how the evolution equation

$$\partial_t B = \nabla \times \frac{((\nabla \times B) \times B) \times B}{|B|^2}, \quad \nabla \cdot B = 0, \tag{1}$$

can be seen as a 'canonical' diffusion equation for 3D divergence-free vector fields, which we claim to be the vectorial counterpart of the standard diffusion equation for scalar density fields, namely

$$\partial_t \rho = \Delta \rho. \tag{2}$$

To justify, this claim, we proceed in three steps:
i) using a classical 'particle' approximation of the heat equation (2) , with a mollification parameter ϵ, we get an evolution equation for the particle positions;
ii) we write this evolution equation, in an abstract form, as a (suitably normalized) gradient flow, with a suitable energy functional, that can be interpreted as a mollified version of the squared L^2 norm of the density field ρ;
iii) we use the same abstract framework in the case of a loop (instead of a set of particles) and the corresponding divergence-free vector field B: we let evolve the loop according to the (normalized) gradient flow of the mollified squared L^2 norm of B;
iv) we express the resulting equation as an evolution equation for B, let the cut-off parameter go to zero and finally obtain the desired equation (1).

Particle approximation of the diffusion equation. It is 'folklore' of particle methods (as been developed in Los Alamos since the 50's, in particular with F.H. Harlow) to look for special solutions of form

$$\rho(t,x) = \frac{1}{N} \sum_{a=1}^{N} \delta_\epsilon(x - X_a(t)), \tag{3}$$

of the mollified diffusion equation

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad v = -\frac{\nabla \rho_\epsilon}{\rho_\epsilon}, \quad \rho_\epsilon(t,x) = \int \delta_\epsilon(x-y)\rho(t,y)dy, \tag{4}$$

where δ_ϵ is a positive smooth approximation of the Dirac mass. We immediately get, for the particle positions $(X_a(t), a = 1, \cdots, N)$, the following self-contained dynamical system

$$\partial_t X_a = -\frac{\sum_{b=1}^{N} \nabla \delta_\epsilon(X_a - X_b)}{\sum_{b=1}^{N} \delta_\epsilon(X_a - X_b)}. \tag{5}$$

(This approximation has been studied by Degond and Mustieles [1]. See also [4, 6, 2] for related topics.) The dynamical system can be written in abstract form

$$\frac{dX}{dt} = -\frac{J'[X]}{j[X]}, \tag{6}$$

where $J'[X]$ denotes the 'functional' derivative of

$$J[X] = \frac{1}{2}\sum_{a,b=1}^{N} \delta_\epsilon(X_a - X_b) \tag{7}$$

and $j[X]$ denotes the 'integrand'

$$j[X]_a = \sum_{b=1}^{N} \delta_\epsilon(X_a - X_b), \quad a = 1, \cdots, N. \tag{8}$$

Notice that $2J[X]/N^2$ can be viewed as a mollified version of the squared L^2 norm of ρ.

Loop evolution. We now consider a loop $a \in \mathbf{R}/\mathbf{Z} \to X_a \in \mathbf{R}^D$ and the corresponding divergence-free vector field

$$B(x) = \int_{\mathbf{R}/\mathbf{Z}} \delta(x - X_a) X_a' da \in \mathbf{R}^D, \tag{9}$$

where $X_a' = dX_a/da$. We introduce the functional

$$J[X] = \frac{1}{2}\int \delta_\epsilon(X_a - X_b) X_a' \cdot X_b' \, dadb. \tag{10}$$

This is just a mollified version of $\int_{\mathbf{R}^D} |B(x)|^2 dx/2$ (which does not make sense for a loop). We also introduce the integrand $j[X]$

$$j[X]_a = \int \delta_\epsilon(X_a - X_b) X_a' \cdot X_b' \, db. \tag{11}$$

Then, after elementary calculations, we get from the abstract equation (6), an evolution equation for B, which, after letting ϵ go to zero, reads:

$$\partial_t B_i + \sum_{j=1}^{N} \partial_j (B_i v_j - B_j v_i) = 0, \tag{12}$$

$$v_i = \frac{1}{|B|^2}\sum_{j=1}^{N} (\partial_j B_i - \partial_i B_j) B_j, \tag{13}$$

which is nothing but (1) in the special case $D = 3$.

Comments.

$$\frac{1}{2}\frac{d}{dt}\int |B|^2 dx + \int \frac{|(\nabla \times B)\times B|^2}{|B|^2}\, dx = 0 \tag{14}$$

formally follows from equation (1). Thus, the diffusion process stops for all fields $B = B(x)$ such that $(\nabla \times B) \times B = 0$. Such fields are special stationary solutions of the 3D Euler equations and seem to play an important role in Turbulence Theory [3] The same property holds true for the variant of equation (1), where there is no denominator $|B|^2$. The later equation was considered by Nishiyama [5] as a way of getting special stationary solutions of the 3D Euler equations.
The analysis of equation (1) seems to us entirely open, as well as its possible interpretation in terms of stochastic processes.

References

[1] P. Degond, F. Mustieles, *A deterministic approximation of diffusion equations using particles,* SIAM J. Sci. Stat. Comput. 11 (1990) 293-310.

[2] A.Figalli, R. Philipowski, *Convergence to the viscous porous medium equation and propagation of chaos,* ALEA Lat. Am. J. Probab. Math. Stat. 4 (2008) 185–203.

[3] U. Frisch, *Turbulence, The legacy of A. N. Kolmogorov* Cambridge University Press, Cambridge, 1995.

[4] P.-L. Lions, S. Mas-Gallic. *Une méthode particulaire déterministe pour des équations diffusives non-linéaires,* C.R. Acad. Sci. Paris Math. 332 (2001) 369-376.

[5] T. Nishiyama, *Construction of the three-dimensional stationary Euler flows from pseudo-advected vorticity equations,* R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. 459 (2003) 2393-2398.

[6] K. Oelschlaeger, *A sequence of integro-differential equations approximating a viscous porous medium equation,* Z. Anal. Anwendungen 20 (2001) 55-91.

A vanishing viscosity approach in damage mechanics

Dorothee Knees

(joint work with R. Rossi and C. Zanini)

We analyze a rate-independent model for damage evolution in elastic bodies. The central quantities are a stored energy functional $\mathcal{E}(t, z, u)$ depending on time t, the displacements u and the damage variable z ($z = 1$ means undamaged, while $z = 0$ stands for maximum damage), and a dissipation functional $\mathcal{R}$, which is assumed to be positively homogeneous of degree one. The damage evolution is governed by the following doubly nonlinear system:

$$0 = \mathrm{D}_u \mathcal{E}(t, z(t), u(t)), \tag{1}$$

$$0 \in \partial \mathcal{R}(\dot{z}(t)) + \mathrm{D}_z \mathcal{E}(t, z(t), u(t)). \tag{2}$$

The first equation describes the quasistatic balance of forces for an elastic body, while the second equation characterizes the evolution law for the damage variable z. The irreversibility of the damage process, i.e. healing is excluded, is enforced by setting $\mathcal{R}(\zeta) = \infty$ if $\zeta > 0$ on some part of the physical body. As a consequence, the subdifferential $\partial \mathcal{R}(0)$ is unbounded, which makes the analysis challenging.

Since in typical damage models the energy $\mathcal{E}$ is not simultaneously convex in the damage variable and the displacements, solutions may have jumps as a function of time, as we will illustrate in the example below. The latter circumstance makes it necessary to introduce a suitable notion of weak solutions.

In the by-now classical concept of global energetic solutions the equations (1)–(2) are reformulated in terms of a global stability criterion and an energy balance that must hold for all t ($\mathcal{U}$, $\mathcal{Z}$ suitable function spaces; special structure of $\mathbf{E}$ due to irreversibility):

S: for all $v \in \mathcal{U}$, $\xi \in \mathcal{Z}$: $\mathcal{E}(t, z(t), u(t)) \le \mathcal{E}(t, \xi, v) + \mathcal{R}(\xi - z(t))$

E: $\mathcal{E}(t, u(t), z(t)) + \mathcal{R}(z(t) - z(0)) = \mathcal{E}(0, z(0), u(0)) + \int_0^t \partial_t \mathcal{E}(\tau, z(\tau), u(\tau)) \mathrm{d}\tau$.

Within this framework the existence of solutions for damage models was investigated eg. in [5]. This point of view allows for solutions that are discontinuous in time. However, as will be shown in the example below, solutions may jump too early and in particular they may jump although a local force balance criterium would predict no motion at all.

As an alternative to the global energetic framework, jump criteria can also be derived by adding an additional viscous dissipation to the problem and then studying the rate-independent limit model as the viscosity tends to zero. This strategy was introduced in the abstract papers [1, 4] and applied to damage models in [3]. There, we use a technique for taking the vanishing viscosity limit, which is based on arc-length reparameterization. In this way, in the limit we obtain a novel formulation for the rate-independent damage model, which highlights the interplay of viscous and rate-independent effects in the jump regime, and provides a better description of the energetic behavior of the system at jumps. In [3] we prove the existence of nondegenerate $\mathcal{Z}$-reparametrized vanishing viscosity solutions of the damage model.

Definition 1. A triple $(\hat{t}, \hat{z}, \hat{u}) \in C_{\mathrm{Lip}}([0, S]; [0, T] \times \mathcal{Z} \times \mathcal{U})$ is a non-degenerate $\mathcal{Z}$-reparametrized vanishing viscosity solution of the damage model, if (1) is valid, and if there exists a Borel function $\lambda : [0, S] \to [0, \infty)$ such that for a.e. $s \in [0, S]$ it holds $\hat{t}'(s) \ge 0$, $\hat{t}'(s) + \|\hat{z}'(s)\|_{\mathcal{Z}} = 1$ and

$$0 \in \partial \mathcal{R}(\hat{z}'(s)) + \lambda(s)\hat{z}'(s) + \mathrm{D}_z \mathcal{E}(\hat{t}(s), \hat{z}(s), \hat{u}(s)), \tag{3}$$

$$0 = \lambda(s)\hat{t}'(s). \tag{4}$$

As can be seen from (4), $\hat{t}' > 0$ on some interval (s_1, s_2) implies $\lambda = 0$ and (3) is just a reparametrized version of the original doubly nonlinear inclusion (2). On the other hand, $\hat{t}' = 0$ on some interval means that the physical time is frozen and an evolution of $\hat{z}$ in this interval induces a jump in the slow external time scale. In this regime, λ might be positive, and hence viscous dissipation is active. It is ongoing research to reformulate the reparametrized model with respect to the original time t and to deduce explicitly the jump criteria. For our analysis it is essential that $\dot{z}$ and the thermodynamically conjugate forces $\mathrm{D}_z \mathcal{E}(t, z, u)$ are in duality so that a chain rule can be applied. Due to the irreversibility constraint on $\mathcal{R}$ and the resulting unboundedness of the setvalued operator $\partial \mathcal{R}(0) : \mathcal{Z} \to \mathcal{Z}^*$

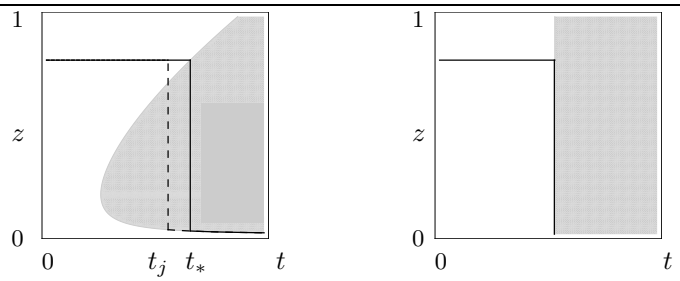

FIGURE 1. Left: Vanishing viscosity solution (solid) and global energetic solution (dashed) for $\mathcal{I}$; Right: Solution for $\mathcal{I}_2$.

the usual energy estimates do not give enough information and refined estimates have to be carried out.

The following zero-dimensional example reveals the differences between the predictions of the vanishing viscosity model and the global energetic model. For $t, z, e \in \mathbb{R}$ let $\mathcal{E}(t,z,e) = \frac{1}{2}g(z)e^2 - te$ with $g(z) = \frac{1}{2}(z^2+\delta)$, and let $\mathcal{R}(\zeta) = \kappa\,|\zeta|$ if $\zeta \le 0$ and $\mathcal{R}(\zeta) = \infty$ otherwise ($\delta = 0.1$, $\kappa = 500$). Then (1) is satisfied if $e(t,z) = t/g(z)$ and we define $\mathcal{I}(t,z) = \min_e \mathcal{E}(t,z,e) \equiv -t^2/2g(z)$. Observe that $\mathcal{I}(t,\cdot)$ is not convex in the second variable although $g(\cdot)$ is. Relations (1)–(2) are equivalent to the following conditions:

$$\dot z(t) \le 0,\ \ \mathrm{D}_z\mathcal{I}(t,z(t)) - \kappa \le 0,\ \ \dot z(t)\big(\mathrm{D}_z\mathcal{I}(t,z(t)) - \kappa\big) = 0.$$

In Fig. 1 (left), in the white region it holds $\mathrm{D}_z\mathcal{I}(t,z) - \kappa < 0$, while in the grey region we have $\mathrm{D}_z\mathcal{I}(t,z) - \kappa > 0$. Starting with the initial value $z_0 = 0.8$, the solution according to Definition 1 (Fig. 1, left, solid line) is constant until time t_*, for which $\mathrm{D}_z\mathcal{I}(t_*, z_0) = \kappa$ for the first time. Then a jump in z takes place, which due to (3)–(4) crosses exactly the region with $\mathrm{D}_z\mathcal{I}(t,z) - \kappa > 0$ (grey). After this jump a slow, rate-independent evolution along the separating line between the two regions (white and grey) takes place.

In contrast to this, in the global energetic framework described by **S** and **E** the energy balance implies that at jump points t_j we have $\int_{z(t_j-)}^{z(t_j+)} \mathrm{D}_z\mathcal{I}(t_j,\xi) - \kappa \mathrm{d}\xi = 0$. Hence, the jump has to cross both, regions where the integrand is negative (white) as well as regions, where the integrand is positive (grey). Due to the global stability **S** this jump takes place as early as possible. After the discontinuity, the solution follows the separating line (see Fig. 1, left, dashed line).

If one replaces g with $g_2(z) := (z/g(1) + (1-z)/g(0))^{-1}$, (g_2 is the harmonic mean of $g(0)$ and $g(1)$ with weight z), the situation changes completely. Now, $\mathcal{I}_2(t,\cdot)$ is linear, and in particular convex, in the second variable. In this case, the global energetic solution and the solution according to Definition 1 coincide, see Fig. 1 (right). The energy with g_2 arises (in 1d) in the model discussed in [2]. At least in 1d the convexity of $\mathcal{I}_2$ explains, why the solutions in [2] are so-called threshold solutions (meaning that damage does not evolve unless the forces reach a certain threshold) although they are defined via a global minimization principle in the spirit of **S** and **E**.

References

[1] M. Efendiev and A. Mielke. *On the rate–independent limit of systems with dry friction and small viscosity.* J. Convex Analysis **13** (2006), 151–167.

[2] A. Garroni and C. Larsen. *Threshold-based quasi-static brittle damage evolution.* Arch. Ration. Mech. Anal. **194(2)** (2009), 585–609.

[3] D. Knees, R. Rossi, and C. Zanini. *A vanishing viscosity approach to a rate-independent damage model.* WIAS-Preprint 1633, Weierstrass Institute, Berlin, (2011).

[4] A. Mielke, R. Rossi, and G. Savaré. *Nonsmooth analysis of doubly nonlinear evolution equations.* Calc. Var. Partial Differential Equations., (2011). to appear.

[5] M. Thomas and A. Mielke. *Damage of nonlinearly elastic materials at small strain - existence and regularity results.* ZAMM Z. Angew. Math. Mech., **90(2)** (2010), 88–112.

On dynamic Griffith fracture

Christopher J. Larsen

We consider a displacement-crack pair $(u(t), \Gamma(t))$ to be a dynamic fracture solution if

i) u follows elastodynamics off of Γ
ii) Γ can only grow in time
iii) the total energy (stored elastic + kinetic + surface energy + work done by loading) is balanced
iv) an additional principle holds, such as a form of maximal dissipation, that can require the crack to grow (otherwise a stationary crack is always a solution).

It is a dynamic Griffith fracture solution if the above hold and the surface energy is (locally) proportional to $\mathcal{H}^{N-1}(\Gamma)$. We note that i)-iii) above are apparently not controversial, and possibilities for iv) based on maximal dissipation were discussed in [2]. Here our focus is on properties i)-iii).

Giving a meaning to i) is not obvious, especially with the spaces required if the only assumption on Γ is a bound on $\mathcal{H}^{N-1}(\Gamma(t))$. The main point of [1] was to give such a meaning, and show, for a given growing crack and initial data, that there exists a u satisfying i).

More precisely, we have the following definitions: u is a weak solution of the damped wave equation on the cracking domain $t \mapsto \Omega \setminus \Gamma(t)$, $t \in [0,T]$, with homogeneous Neumann boundary conditions if

$$u \in H^1(0,T;V) \cap W^{1,\infty}(0,T;L^2), \tag{1}$$

$$\text{for every } t \in [0,T] \text{ we have } u(t) \in V_t, \tag{2}$$

$$\text{for every } s \in [0,T) \text{ we have } u \in W^{2,\infty}(s,T;V_s^*), \tag{3}$$

$$\sup_{s\in[0,T)} \|\ddot{u}\|_{L^\infty(s,T;V_s^*)} < +\infty, \tag{4}$$

and for a.e. $t \in [0,T]$

$$\langle \ddot{u}(t), \phi\rangle_t + \langle \nabla u(t) + \gamma \nabla \dot{u}(t), \nabla\phi\rangle_{L^2} = \langle f(t), \phi\rangle_{L^2} \quad \text{for every } \phi \in V_t, \tag{5}$$

where $V := V_T$, V_t^* is the dual of V_t with pairing $\langle \cdot, \cdot \rangle_t$, consistent with the L^2 pairing. $V_t := \{v \in GSBV(\Omega) \cap L^2(\Omega) : \nabla v \in L^2(\Omega; \mathbb{R}^N), S_v \subset \Gamma(t)\}$.

u is a weak solution of the (undamped) wave equation on the cracking domain $t \mapsto \Omega \setminus \Gamma(t)$ if

$$u \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2), \tag{6}$$

$$\text{for every } t \in [0,T] \text{ we have } u(t) \in V_t, \tag{7}$$

$$\text{for every } s \in [0,T) \text{ we have } u \in W^{2,\infty}(s,T;V_s^*), \tag{8}$$

$$\sup_{s \in [0,T)} \|\ddot{u}\|_{L^\infty(s,T;V_s^*)} < +\infty, \tag{9}$$

and for a.e. $t \in [0,T]$

$$\langle \ddot{u}(t), \phi \rangle_t + \langle \nabla u(t), \nabla \phi \rangle_{L^2} = \langle f(t), \phi \rangle_{L^2} \quad \text{for every } \phi \in V_t. \tag{10}$$

The main difference between (1) and (6) is that $\nabla \dot{u}(t) \notin L^2$ without damping.

We then prove (Theorems 3.2 and 4.2) the following. Given suitable initial data $u^{(0)}, u^{(1)}$, there exist solutions for both, plus, in the damped case, there is energy balance:

$$\frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{2}\|\dot{u}(t)\|_{L^2}^2 + \gamma \int_0^t \|\nabla \dot{u}(\tau)\|_{L^2}^2 d\tau - \int_0^t \langle f(\tau), \dot{u}(\tau) \rangle_{L^2} d\tau$$

is constant, and therefore we have uniqueness.

Note that this energy balance is undesirable, since it rules out the balance of the total energy, which includes the surface energy. The main future interest is therefore in the undamped equation (or perhaps, a damped equation with a different kind of damping).

Outline of existence proof: We define u_n^i for $i = -1, 0, ..., n$ inductively by the following: First,

$$u_n^0 := u^{(0)}, \quad u_n^{-1} := u^{(0)} - \tau_n u^{(1)}; \tag{11}$$

then, for $i = 0, 1, \ldots, n-1$, the function u_n^{i+1} is the minimizer in $V_{t_n^{i+1}}$ of

$$u \mapsto \left\| \frac{u - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n} \right\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \frac{\gamma}{\tau_n}\|\nabla u - \nabla u_n^i\|_{L^2}^2.$$

It follows that we have

$$\left\langle \frac{u_n^{i+1} - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n}, \frac{\phi}{\tau_n} \right\rangle_{L^2} + \langle \nabla u_n^{i+1}, \nabla \phi \rangle_{L^2} + \frac{\gamma}{\tau_n} \langle \nabla u_n^{i+1} - \nabla u_n^i, \nabla \phi \rangle_{L^2} = 0 \tag{12}$$

for every $\phi \in V_{t_n^{i+1}}$. We can take $\phi = u_n^{i+1} - u_n^i$ and eventually get:

for every $t \in (t_n^i, t_n^{i+1})$:

$$\|\dot{u}_n(t)\|_{L^2}^2 + \|\nabla u_n(t_n^{i+1})\|_{L^2}^2 + \tau_n \int_0^{t_n^{i+1}} \|\dot{v}_n(t)\|_{L^2}^2 \, dt + \tau_n \int_0^{t_n^{i+1}} \|\nabla \dot{u}_n(t)\|_{L^2}^2 dt$$

$$+2\gamma \int_0^{t_n^{i+1}} \|\nabla \dot{u}_n(t)\|_{L^2}^2 \, dt = \|u^{(1)}\|_{L^2}^2 + \|\nabla u^{(0)}\|_{L^2}^2, \tag{13}$$

where $u_n(t)$ is the affine interpolation between u_n^i and u_n^{i+1}, and $v_n(t)$ is the affine interpolation between $\frac{u_n^i - u_n^{i-1}}{\tau_n}$ and $\frac{u_n^{i+1} - u_n^i}{\tau_n}$. This gives bounds we need for compactness.

Note that this holds also with $\gamma = 0$. We also get that for all $t \in (t_n^i, t_n^{i+1})$,

$$\langle \dot{v}_n(t), \phi \rangle_{L^2} + \langle \nabla \tilde{u}_n(t) + \gamma \nabla \dot{u}_n(t), \nabla \phi \rangle_{L^2} = 0 \tag{14}$$

for every $\phi \in V_{t_n^{i+1}}$, where $\tilde{u}_n$ is piece-wise constant in time. We then show that $\dot{v}_n \to \ddot{u}, \nabla \tilde{u}_n \to \nabla u$, and $\nabla \dot{u}_n \to \nabla \dot{u}$, in the appropriate senses. □

References

[1] G. Dal Maso, C. J. Larsen, *Existence for wave equations on domains with arbitrary growing cracks*, Rend. Lincei Mat. Appl. **22** (2011), 387–408.

[2] C. J. Larsen, *Models for dynamic fracture based on Griffith's criterion*, in IUTAM Symposium on Variational Concepts with Applications to the Mechanics of Materials (Klaus Hackl, ed.), Springer, 2010, 131–140.

Quantum Navier-Stokes equations: kinetic and Lagrangian approaches

Ansgar Jüngel

(joint work with P. Fuchs and M. von Renesse)

We review some results obtained for a certain class of quantum fluid models, namely the *quantum Navier-Stokes equations*. Quantum fluid models have the advantage that quantum effects are included by correction terms, which allows, e.g., for a hybrid classical-quantum modeling. Isothermal quantum Navier-Stokes equations have been formally derived by Brull and Méhats [1] from the Wigner-BGK equation (named after Bhatagnar, Gross, and Krook) by a Chapman-Enskog expansion around the quantum equilibrium and by expanding the resulting nonlocal equations in terms of powers of the scaled Planck constant. The equations for the particle density n and the velocity u read as

(1) $\partial_t n + \operatorname{div}(nu) = 0, \quad x \in \mathbb{R}^d, \ t > 0,$

(2) $\partial_t(nu) + \operatorname{div}(nu \otimes u) + \nabla p(n) + n\nabla V(x) - \dfrac{\varepsilon^2}{6} n \nabla \left(\dfrac{\Delta \sqrt{n}}{\sqrt{n}} \right) = \nu \operatorname{div}(nD(u)),$

where $p(n) = n^\gamma$ ($\gamma \geq 1$) is the pressure, $V(x)$ is a potential, and $D(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ is the symmetric velocity gradient. The (scaled) physical parameters are the Planck constant $\varepsilon > 0$ and the viscosity constant $\nu > 0$ (related to the mean free path in the Wigner-BGK model). Finally, initial conditions for n and nu

are prescribed, $n(\cdot,0)=n_0$, $(nu)(\cdot,0)=n_0u_0$ in $\mathbb{R}^d$. Non-isothermal quantum Navier-Stokes equations have been derived in [4].

When equations (1)-(2) are considered on the d-dimensional torus $\mathbb{T}^d$ (to avoid technicalities due to boundary conditions), the global-in-time existence of weak solutions has been proved [3]. The key idea of the proof is the discovery that the system possesses two energy functionals. Indeed, the standard energy

$$E_1=\int_{\mathbb{T}^d}\Big(\frac{n}{2}|u|^2+H(n)+\frac{\varepsilon^2}{2}|\nabla\sqrt{n}|^2\Big)dx,$$

where $H(n)=n^\gamma/(\gamma-1)$ if $\gamma>1$ and $H(n)=n(\log n-1)$ if $\gamma=1$, is not sufficient for the existence analysis since it gives only an H^1 estimate for $\sqrt{n}$. On the other hand, using the new velocity variable $w=v+\nu\nabla\log n$, the energy dissipation of the second functional

$$E_2=\int_{\mathbb{T}^d}\Big(\frac{n}{2}|w|^2+H(n)+\frac{1}{4}\Big(\frac{\varepsilon^2}{3}-\nu^2\Big)|\nabla\sqrt{n}|^2\Big)dx$$

provides H^2 estimates for $\sqrt{n}$ if $\varepsilon^2/3>\nu^2$. This forms the basis of the existence analysis, which is carried out by using a Faedo-Galerkin approximation and weak compactness tools, see [3] for details.

One may ask *why the system possesses two energies*. In order to investigate this question, we have formulated a Lagrangian mechanics theory on the space of probability measures [2]. Given a classical Lagrangian function $L(q,\dot q)$, a lifted Lagrangian $\mathcal{L}(\mu,\eta)$ is defined by the infimum of the integrals $\int_{\mathbb{R}^d}L(x,u(x))\mu(dx)$ over the set of velocities $u\in C^\infty(\mathbb{R}^d;\mathbb{R}^d)$ satisfying $\eta+\operatorname{div}(\mu u)=0$. Here, μ is a probability measure (corresponding to the particle density n) and η is an element of the tangent space at μ (corresponding to $\partial_t n$). For example, the kinetic energy $L(q,\dot q)=\frac12|\dot q|^2$ gives Otto's Riemannian tensor on the tangent bundle, $\mathcal{L}(\mu,\eta)=\int_{\mathbb{R}^d}|\nabla S(x)|^2\mu(dx)$, where $S:=\Delta_\mu^{-1}\eta$ and $\Delta_\mu S:=\operatorname{div}(\mu\nabla S)$ in $\mathbb{R}^d$.

The quantum Navier-Stokes equations are recovered as a (formal) critical point of the dissipative Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\eta}-\frac{\partial\mathcal{L}}{\partial\mu}+\frac{\partial\mathcal{D}}{\partial\eta}=0$$

for the quantum Lagrangian

$$\mathcal{L}(\mu,\eta)=\int_{\mathbb{R}^d}\Big(\frac12|\nabla S|^2-U(\mu)-V(x)-\frac{\varepsilon^2}{4}|\nabla\log\mu|^2\Big)\mu(dx),$$

where $U(\mu)$ denotes the internal energy, and the dissipation potential is $\mathcal{D}(\mu,\eta)=(\nu/2)\int_{\mathbb{R}^d}|\nabla u|^2\mu(dx)$ with $u=\nabla S$. More precisely, it is shown in [2] that a smooth curve $\mu(t)$ satisfies the above Euler-Lagrange equation if and only if $(\mu,\mu u)$ solves the quantum Navier-Stokes equations (1)-(2) with $n=\mu$ and the viscosity replaced by $\nu\mu\nabla\Delta_\mu^{-1}(\nabla^2:(\mu\nabla u))$. Projecting this expression on the space of curl-free fields, we recover the viscosity term in (2).

Now, we come back to the question raised above. We will give a partial answer. In classical Noether theory, the Noether current for the transformation $t\mapsto t+\delta t$,

$q \mapsto q + \delta q$ is defined by $J = \delta t(\dot{q}\partial L/\partial \dot{q} - L) - \delta q \partial L/\partial \dot{q}$. This motivates the *lifted Noether current* on the space of probability measures:

$$\mathcal{J}(\mu, \eta) = \delta t \frac{\partial \mathcal{L}}{\partial \eta}(\mu, \eta)\eta - \delta t \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \eta}(\mu, \eta)\delta\mu.$$

As expected, under time shifts $\delta t = 1$, $\delta\mu = 0$, the Noether current equals the classical energy functional E_1. Surprisingly, with the transformation $\delta t = 1$, $\delta\mu = \nu\Delta\mu$, the corresponding Noether current equals the second energy E_2. This gives rise to a number of *open problems*:

(1) Are there other transformations leading to new energy functionals?
(2) How can we develop a dissipative Noether theory on the space of measures?
(3) Can this theory be applied to classical fluid dynamics? Which fluid dynamical models possess several energies?
(4) Is this a first step to devise an optimal transportation formulation of quantum mechanics?

References

[1] S. Brull and F. Méhats, *Derivation of viscous correction terms for the isothermal quantum Euler model*, Z. Angew. Math. Mech. **90** (2010), 219–230.
[2] P. Fuchs, A. Jüngel, and M. von Renesse, *On the Lagrangian structure of quantum fluid models*, preprint, 2011. `http://www.juengel.at.vu.`
[3] A. Jüngel, *Global weak solutions to compressible Navier-Stokes equations for quantum fluids*, SIAM J. Math. Anal. **42** (2010), 1025–1045.
[4] A. Jüngel and J.-P. Milisic, *Full compressible Navier-Stokes equations for quantum fluids: derivation and numerical solution*, Kinetic Rel. Models **4** (2011), 785–807.

On the coarsening rates in demixing binary viscous liquids

Christian Seis
(joint work with Y. Brenier, F. Otto)

We consider a demixing process of a binary viscous liquid. Thermodynamics favors the separation of the mixture into two domains of the two equilibrium volume fractions. The average size of these domains grows during the evolution, and thus, the domain morphology coarsens. We report on a work [1] that investigates the rates at which this coarsening proceeds.

The demixing process can be described in terms of the order parameter m, a measure for the local composition of the mixture. In viscous mixtures, material transport is due to diffusion, i.e., the cross-motion of unlike particles, and convection, i.e., the transport by the hydrodynamic bulk flow. Therefore, the evolution equation comes as a convection-diffusion equation for m:

$$\partial_t m + u \cdot \nabla m - \lambda\Delta\mu \ = \ 0,$$

where μ is the chemical potential defined by $\mu = -\Delta m - 2m(1 - m^2)$. The parameter λ measures the strength of diffusion compared to convection. The

liquid velocity u obeys a Stokes equation:

$$\begin{aligned} -\Delta u + \nabla p &= -m\nabla\mu \\ \nabla \cdot u &= 0, \end{aligned}$$

where p is the hydrodynamic pressure. The system is supplemented by periodic boundary conditions with period cell $[0,\Lambda]^d$, where $\Lambda \gg 1$, i.e., the period length is supposed to be much larger than the thickness of the interfacial layer between the domains.

This dynamical system has a formal gradient flow interpretation, that is, the evolution follows the steepest descent in an energy landscape, $\partial_t m + \nabla E = 0$. The energy functional is given by the Ginzburg–Landau energy density

$$E \;=\; \frac{1}{\Lambda^d}\int \frac{1}{2}|\nabla m|^2 + \frac{1}{2}(1-m^2)^2\,dx,$$

and the metric tensor is

$$\begin{aligned} &g_m(\delta m,\delta m) \\ (1)\qquad &= \inf\left\{\frac{1}{\Lambda^d}\int \frac{1}{\lambda}|j|^2 + |Du|^2\;dx \;\middle|\; \delta m + \nabla\cdot j + u\cdot\nabla m = 0,\ \nabla\cdot u = 0\right\}. \end{aligned}$$

In the following we will consider convective transport alone, and therefore we set $\lambda = 0$ (which enforces $j = 0$).

In real experiments and numerical simulations it is observed that the typical size ℓ of the domains grows during the demixing process according to the power law

$$\ell \;\sim\; t, \tag{2}$$

when t denotes time. A first heuristic explanation of this coarsening law is due to Siggia [4].

First rigorous estimates on coarsening rates were derived by Kohn and Otto in [2] for two Cahn–Hilliard equations. The authors introduce a method that is based on the gradient flow structure of the dynamics and produce lower bounds on the energy. Since the energy scales, at least heuristically, as an inverse length, and assuming that there is only one length scale present in the dynamics, lower bounds on the energy can be interpreted as upper bounds on the coarsening rates.

For convenience, we suppose that $m \in \{\pm 1\}$, and E is proportional to the quotient of the interfacial area and the system volume. Following the method of [2], we introduce a function L that can be considered as a proxy to the geodesic distance function induced by the metric tensor (1) (with $\lambda = 0$). Indeed, choosing

$$L \;=\; \inf_{\pi}\left\{\frac{1}{\Lambda^d}\int\int \ln(|x-y|+1)\,d\pi(x,y)\;\middle|\; \int d\pi(\cdot,y) = m+1,\ \int d\pi(x,\cdot) = 1\right\},$$

we establish the dissipation inequality

$$\left(\frac{dL}{dt}\right)^2 \;\lesssim\; \left(-\frac{dE}{dt}\right)$$

as a consequence of the gradient flow structure of the dynamics. The function L, a so-called Monge–Kantorovich–Rubinstein distance, measures the optimal transportation cost for transferring the configuration "1" into the configuration "$m+1$" subject to the logarithmic cost function, cf. [5]. Moreover, E and L are dual in the sense that

$$L \gtrsim \ln \frac{1}{E}.$$

The heart of the Kohn–Otto method is an ODE argument that translates the information on how fast the energy decreases as a function of distance into information on how fast the energy decreases as a function of time:

Theorem 1.

$$\int_0^T E(t)\,dt \gtrsim \ln T \quad \textit{provided that } \ln T \gg L(0).$$

In fact, this result is a weak, one-sided version (2) with $E \sim \frac{1}{\ell}$. Notice that due to the existence of ungeneric, slowly evolving solutions, only upper bounds on coarsening rates can be established via a priori estimates. Coarsening rates for the full model that allows for diffusive *and* convective transport is treated in [3].

References

[1] Brenier, Y., Otto, F., and Seis, C. Upper bound on coarsening rates in demixing binary viscous liquids *SIAM J. Math. Anal.* **43(1)**: 114-134, 2011.

[2] Kohn, R. V. and Otto, F. Upper Bounds on Coarsening Rates. *Comm. in Math. Phys.* **229(3)**: 375-395, 2002.

[3] Otto, F, Seis, C., and Slepčev, D. Crossover in the coarsening rates in demixing binary viscous fluids. *In preparation.*

[4] Siggia, E. D. Late stages of spinodal decomposition in binary mixtures. *Phys. Rev. A* **20**: 595-605, 1979.

[5] Villani, C. *Topics in Optimal Transportation.* Volume 58 of *Graduate Studies in Mathematics.* Amer. Math. Soc., Providence, RI, 2003.

Understanding the origins of Wasserstein gradient flows

Mark A. Peletier

'Variational Modelling' is an appealing way of modelling real-world systems. In the case of gradient flows, one models a system by postulating a driving functional (energy or entropy) and a dissipation mechanism. These two choices together determine the evolution, and especially for more complex systems they often give a simpler modelling route than other methods. Also, thermodynamic consistency is usually automatically satisfied.

But Wasserstein gradient flows have components whose modelling interpretations are obscure. The Wasserstein metric itself, but also the entropy and the free energies that often drive such systems do not have a straightforward modelling interpretation in the context of evolving systems.

In this talk I showed how the role of entropy, free energy, and of the Wasserstein metric and Wasserstein gradient flows can be understood, and should be

understood, by connecting the deterministic gradient flows with large deviations of stochastic particle systems. The central observation is that Wasserstein gradient flows typically describe the evolution of densities ρ, which can be thought of as the limit of empirical measures

$$L_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i} \tag{1}$$

as the number n of particles tends to infinity.

With this observation one can understand a number of properties.

(1) The entropy $\mathrm{Ent}(\rho) := \int \rho \log \rho \, dx$ arises from the *indistinguishibility* of the particles in an empirical measure. This is the property that renumbering the particles leaves the empirical measure unchanged. The degeneracy that follows from this indistinguishibility can be shown, by a counting argument and Stirling's formula, to lead to the entropy [1]. The mathematical expression of this property is the fact that if we draw n particles X_i from a distribution μ on a state space $\mathcal{X}$, we have the large-deviation result (by Sanov)

$$\mathrm{Prob}(L_n \approx \rho) \sim \exp -nI(\rho) \qquad \text{as } n \to \infty,$$

where I has the characterization in terms of the relative entropy $\mathcal{H}$,

$$I(\rho) = \mathcal{H}(\rho|\mu) := \begin{cases} \displaystyle\int \frac{\mathrm{d}\rho}{\mathrm{d}\mu} \log \frac{\mathrm{d}\rho}{\mathrm{d}\mu}\, d\mu & \text{if } \rho \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

This large-deviation result shows how the relative entropy $\mathcal{H}(\rho|\mu)$ characterizes the (im)probability of observing $L_n \approx \rho$.

(2) A free energy of the form $\mathcal{H}(\rho|\mu) + \int_{\mathcal{X}} e(x)\, \rho(dx)$ can similarly be interpreted as a large-deviation rate functional. It arises from the coupling of a system as above to a heat bath. A heat bath, in this setup, is represented by a second system $\mathcal{Y}$ with distribution ν, and the two are coupled by choosing energy functions $e : \mathcal{X} \to \mathbb{R}$ and $e_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{R}$ and imposing a prescribed total energy $\overline{E}$:

$$\mathrm{Prob}\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{X_i} \approx \rho \;\middle|\; \frac{1}{n}\sum_{i=1}^{n} e(X_i) + \frac{1}{nN}\sum_{j=1}^{nN} e_{\mathcal{Y}}(Y_j) = \overline{E} \right) \sim \exp -nJ(\rho) \qquad \text{as } n \to \infty \text{ and } N \to \infty$$

where

$$J(\rho) = \mathcal{H}(\rho|\mu) + \frac{1}{kT}\int_{\mathcal{X}} e(x)\, \rho(dx) + \text{constant},$$

where the constant is chosen such that $\inf J = 0$, and kT is defined by a characterization in terms of the properties of the heat bath. In this setup we take n particles from $\mathcal{X}$, and nN particles from $\mathcal{Y}$, where N is large; this represents the assumption that the heat bath is 'large' with respect to

the system $\mathcal{X}$. The limit $N \to \infty$ allows us to characterize the influence of the heat bath by the single constant kT. The whole argument can be found in [3].

(3) The Wasserstein distance $d(\rho_0, \rho_1)^2$ arises as the characterization of the mobility of empirical measures of Brownian particles. If X_i are Brownian particles in $\mathbb{R}^d$, then L_n defined in (1) is a time-dependent, random measure $t \mapsto L_n(t)$. We then have [2, 4]

$$\mathrm{Prob}\Big(L_n(t) \approx \rho_t \big| L_n(0) \approx \rho_0\Big) \sim \exp -nK_t(\rho_t; \rho_0) \qquad \text{as } n \to \infty,$$

where

$$tK_t(\,\cdot\,;\rho_0) \xrightarrow{\Gamma} \frac{1}{4} d(\rho_0, \,\cdot\,)^2 \qquad \text{as } t \downarrow 0.$$

For small times, therefore, *the Wasserstein metric characterizes the mobility of empirical measures of Brownian particles*, in the same way as the entropy characterizes empirical measures of particles drawn from a distribution.

(4) Finally, the result above can be strengthened [1] to

$$K_t(\,\cdot\,;\rho_0) - \frac{1}{4t} d(\rho_0, \,\cdot\,)^2 \xrightarrow{\Gamma} \frac{1}{2}\mathrm{Ent}(\,\cdot\,) - \frac{1}{2}\mathrm{Ent}(\rho_0) \qquad \text{as } t \downarrow 0.$$

This result suggests that

$$K_t(\rho;\rho_0) \approx R_t(\rho;\rho_0) := \frac{1}{4t} d(\rho_0, \rho)^2 + \frac{1}{2}\mathrm{Ent}(\rho) - \frac{1}{2}\mathrm{Ent}(\rho_0) \qquad \text{as } t \downarrow 0,$$

and R_t is the well-known JKO functional that defines a time-discrete approximation of the Wasserstein gradient flow of entropy, given for time step h by

$$\text{For each } k, \text{ define } \rho^{k+1} \text{ by } \rho^{k+1} \in \operatorname*{argmin}_{\rho} R_h(\rho; \rho^k).$$

Each of the large-deviation results above connects the probabilistic behaviour of empirical measures of a system of stochastic particles on one hand with deterministic objects that appear in the modelling of Wasserstein gradient flows on the other. This connection provides the explanation *why* they appear, how their appearance (and their structure) depends on assumptions that are made, and how they might be generalized to more complex systems. In this way they provide a valuable basis for the modelling of Wasserstein gradient flows.

References

[1] S. Adams, N. Dirr, M. A. Peletier, and J. Zimmer. From a large-deviations principle to the Wasserstein gradient flow: A new micro-macro passage. *Communications in Mathematical Physics*, 307:791–815, 2011.

[2] C. Léonard. A large deviation approach to optimal transport. *Arxiv preprint arXiv:0710.1461*, 2007.

[3] M. A Peletier. *Energies, gradient flows, and large deviations: a modelling point of view*, 2011. Lecture notes for the School 'Mathematical Principles for and Advances in Continuum Mechanics', Pisa, 2011; to appear.

[4] M. A. Peletier and M. Renger. Variational formulation of the Fokker-Planck equation with decay: A particle approach. *Arxiv preprint arXiv:1108.3181*, 2011.

Radial Stability/Instability for Repulsive-Attractive Potentials

José A. Carrillo
(joint work with D. Balagué, T. Laurent, G. Raoul)

Nonlocal interaction equations with repulsive-attractive radial potentials were discussed. In their simplest form, nonlocal interaction equations can be written as

$$\frac{d\mu}{dt} + \operatorname{div}(\mu v) = 0 \quad , \quad v = -\nabla W * \mu \tag{1}$$

where $\mu(t,x) = \mu_t(x)$ is the probability or mass density of particles at time t and at location $x \in \mathbb{R}^N$, $W : \mathbb{R}^N \to \mathbb{R}$ is the interaction potential and $v(t,x)$ is the velocity of the particles. We always assume that the interaction potential $W(x) = k(|x|)$ is radial and C^2- or C^3-smooth away from the origin, depending on the results. Typically the potentials we will consider have a singularity at the origin.

Such equations describe the evolution of a continuum density of particles in which they repulse each other in the short range and attract each other in the long range. We proved that under some conditions on the potential, radially symmetric solutions converge exponentially fast in some transport distance toward a spherical shell stationary state. Otherwise we proved that it is not possible for a radially symmetric solution to converge weakly toward the spherical shell stationary state. We also investigated under which condition it is possible for a non-radially symmetric solution to converge toward a singular stationary state supported on a general hypersurface. Finally we provided a detailed analysis of the specific case of the repulsive-attractive power law potentials.

More precisely, we focused primarily on proving rigorous results about the convergence of radially symmetric solutions toward spherical shell stationary states in multi-dimensions. The spherical shell of radius R, denoted δ_R, is the probability measure which is uniformly distributed on the sphere $\partial B(0,R) = \{x \in \mathbb{R}^N : |x| = R\}$. Given a repulsive-attractive radial potential whose attractive force does not decay too fast at infinity, there always exists an $R > 0$ so that the spherical shell of radius R is a stationary state.

It is classical that the equation (1) is a gradient flow of the interaction energy

$$E[\mu] = \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) d\mu(x) d\mu(y)$$

with respect to the euclidean Wasserstein distance. Thus, stable steady states of (1) are expected to be local minimizers of the interaction energy. Simple energetic arguments will show that in order for the spherical shell of radius R to be a local minimum of the interaction energy, it is necessary that the potential W satisfies:

(C0) Repulsive-Attractive Balance: $\omega(R,R) = 0$,
(C1) Fattening Stability: $\partial_1 \omega(R,R) \leq 0$,

(C2) Shifting Stability: $\partial_1\omega(R,R) + \partial_2\omega(R,R) \leq 0$,

where the function $\omega : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ is defined by

$$\omega(r,\eta) = -\frac{1}{\sigma_N}\int_{\partial B(0,1)} \nabla W(re_1 - \eta y)\cdot e_1\, d\sigma(y), \tag{2}$$

σ_N is the area of the unit ball in $\mathbb{R}^N$, e_1 is the first vector of the canonical basis of $\mathbb{R}^N$, $d\sigma$ denotes the volume element of the manifold where the integral is performed and $\mathbb{R}^2_+ = (0,+\infty)\times(0,+\infty)$. Condition **(C0)** simply guarantees that the spherical shell δ_R is a critical point of the interaction energy. We showed that if condition **(C1)** is not satisfied then it is energetically favorable to split the spherical shell into two spherical shells. Heuristically this indicate that the density of particles, rather than remaining on the sphere, is going to expand and occupy a domain in $\mathbb{R}^N$ of positive Lebesgue measure. If condition **(C1)** is not satisfied we say that the "fattening instability" holds. It can be easily checked that if $\omega(R,R) = 0$, then $\partial_1\omega(R,R)$ is simply the value of the divergence of the velocity field on the sphere of radius R. So the fattening instability corresponds to an expanding velocity field on the support of the steady state. We also showed that if condition **(C2)** is not satisfied it is energetically favorable to increase or decrease the radius of the spherical shell. This instability is referred as the "shift instability". These conditions then dictate the radial nonlinear stability/instability of the spherical shells.

A two-scale proof of the Eyring-Kramers formula

Georg Menz
(joint work with André Schlichting)

We apply a combination of the two-scale approach [4] and a transportation technique [2] to give a new proof of the Eyring-Kramers formula. Let us consider a diffusion in a potential landscape given by a smooth Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}$ in the regime of small noise $\varepsilon \ll 1$. The generator of the diffusion is given by

$$L = \varepsilon\Delta - \nabla H\cdot\nabla.$$

The associated Dirichlet form is given by

$$\mathcal{E}(f) = \int (-Lf) f\; d\mu = \varepsilon\int |\nabla f|^2\; d\mu.$$

Under some growth assumptions on H, there exists an equilibrium measure of the according stochastic process. It is called *Gibbs measure* and is given by

$$\mu(dx) = \frac{1}{Z_\mu}\exp\left(-\frac{H(x)}{\varepsilon}\right)\; dx \qquad \text{with} \qquad Z_\mu = \int \exp\left(-\frac{H(x)}{\varepsilon}\right)\; dx.$$

The Eyring-Kramers formula asymptotically determines the Spectral Gap (SG) of the operator L in the regime of small noise $\varepsilon \ll 1$. The SG constant of L is determined by the largest constant $\varrho > 0$ such that for any function f

$$\operatorname{var}_\mu(f) \le \frac{\varepsilon}{\varrho} \int |\nabla f|^2 d\mu. \tag{1}$$

For convenience, we state the Eyring-Kramers formula in a simplified situation:

Proposition (Eyring-Kramers formula [3])**.** *Assume that*

- *H has only two local minima at m_0 and m_1 such that $H(m_0) < H(m_1)$;*
- *s denotes the smallest saddle between m_0 and m_1;*
- *λ^- denotes the negative eigenvalue of* $\operatorname{Hess} H(s)$.

Then the SG constant ϱ of the operator L is given by

$$\frac{1}{\varrho} \le \frac{2\pi}{|\lambda^-|} \frac{\sqrt{|\det \operatorname{Hess} H(s)|}}{\sqrt{|\det \operatorname{Hess} H(m_1)|}} \exp\left(-\frac{H(s) - H(m_1)}{\varepsilon}\right) \left(1 + O(\sqrt{\varepsilon}|\ln \varepsilon|^{\frac{3}{2}})\right).$$

The last theorem states that for small noise $\varepsilon \ll 1$ the SG constant ϱ is essentially determined by the saddle height $(H(s) - H(m_1))$ (cf. Figure 1). The first rigorous proof of the Eyring-Kramers formula in full generality was given by Bovier, Gayrard, and Klein [1] using potential theory. Slightly later, a different proof was given by Helffer, Klein, and Nier [5] via the Witten complex approach.

Now, let us sketch the main idea of our new proof: Let μ_i, $i = 1, 2$, denote the restriction of the Gibbs measure μ to the domain of attraction of the local minima

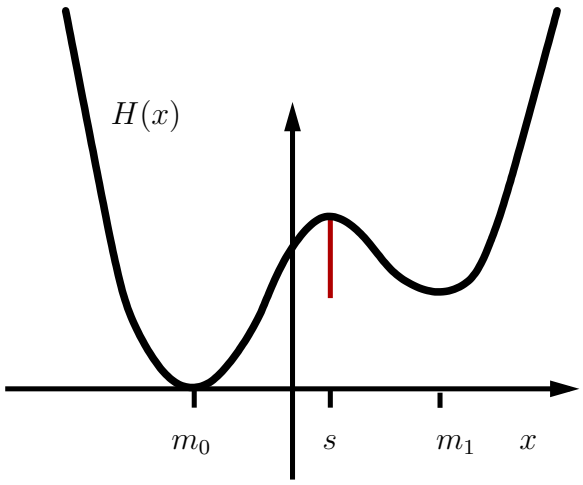

FIGURE 1. Example of an one-dimensional Hamiltonian H

m_i respectively i.e.

$$\mu_i(dx) \;=\; \frac{1}{Z_i}\, 1_{\Omega_i}\, \exp\left(-\frac{H(x)}{\varepsilon}\right) dx,$$

where

$$Z_i = \int_{\Omega_i} \exp\left(-\frac{H(x)}{\varepsilon}\right) dx \quad \text{and} \quad \Omega_i = \left\{x \in \mathbb{R}^N;\ \lim_{t\to\infty} y_t = m_i,\ y_0 = x\right\}.$$

As in the two scale-approach [4], the starting point of our proof is the decomposition of $\mathrm{var}_\mu(f)$ into local variances with respect to μ_0 and μ_1 and into the variance of a Bernoulli variable i.e.

$$(2)\quad \mathrm{var}_\mu(f) = \frac{Z_0}{Z_\mu}\ \mathrm{var}_{\mu_0}(f) \;+\; \frac{Z_1}{Z_\mu}\ \mathrm{var}_{\mu_1}(f) \;+\; \frac{Z_0}{Z_\mu}\frac{Z_1}{Z_\mu}\left(\int f d\mu_0 - \int f d\mu_1\right)^2.$$

The first two terms on the right hand side of (2) are estimated by an application of the SG for the restricted measures μ_0 and μ_1, The SG constants are of order 1. Heuristically, this fact seems to be plausible because there are no metastabilities on the sets Ω_1 and Ω_2. However, the rigorous proof of this fact in higher dimensions is technically challenging because of the lack of convexity of H. The main contribution to the SG constant comes from the third term on the right hand side of (2). Motivated by the transportation technique of Chafï and Malrieu [2], the third term is represented by using a transport Φ_t between μ_0 and μ_1 as

$$\int f d\mu_0 - \int f d\mu_1 = -\int_0^1 \int \frac{d}{dt} f \circ \Phi_t\ d\mu_0\ dt = -\int_0^1 \int \nabla f \cdot \left(\dot\Phi_t \circ \Phi^{-1}\right)\ d\mu_t\ dt,$$

where μ_t denotes the push forward $(\Phi_t)_\# \mu_0$. With some basic intermediate steps, this representation yields the estimate

$$\left(\int f d\mu_0 - \int f d\mu_1\right)^2 \le \int \left(\int_0^1 |\dot\Phi_t \circ \Phi^{-1}|\ \frac{d\mu_t}{d\mu}\ dt\right)^2 d\mu \quad \int |\nabla f|^2\ d\mu.$$

Now, the last step is to estimate the weighted transportation cost on the right hand side of the last inequality. For small noise $\varepsilon \ll 1$, it suffices to consider truncated Gaussian measures around the local minima m_0 and m_1 instead of the restricted measures μ_0 and μ_1. This simplifies the estimation of the weighted transportation cost yielding the Eyring-Kramers formula after an optimization procedure.

A nice feature of this proof is that it replicates the behavior of the corresponding stochastic process: The fast convergence to local minima is expressed by the good local SG estimate of the restricted measures μ_0 and μ_1, whereas the main contribution to the overall SG constant comes from a Markov chain jumping from one local minima to another. Additionally, this approach has the perspective of being applicable to the LSI. Our argument for the local SG also seems to be fruitful: It may be used to show that the asymmetric Mexican hat does not freeze at low temperatures.

References

[1] Bovier, A. and Gayrard, V. and Klein, M., *Metastability in reversible diffusion processes. II. Precise asymptotics for small eigenvalues*, Journal of the European Mathematical Society (JEMS), **7** (2005), 69–99.

[2] Chafaï, D. and Malrieu, F., *On fine properties of mixtures with respect to concentration of measure and Sobolev type inequalities*, Ann. Inst. H. Poincaré Probab. Statist., **46** (2010), 72–96.

[3] Glasstone, S. and Laidler, K. J. and Eyring, H., *The Theory of Rate Processes*, McGraw-Hill, New York, (1941).

[4] Grunewald, N. and Otto, F. and Villani, C. and Westdickenberg, M., *A two-scale approach to logarithmic Sobolev inequalities and the hydrodynamic limit*, Ann. Inst. H. Poincaré Probab. Statist., **45** (2009), 302–351.

[5] Helffer, B. and Klein, M. and Nier, F., *Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach*, Mat. Contemp., **26** (2004), 41–85.

Motion of vortices in ferromagnets

Matthias Kurzke

(joint work with C. Melcher, R. Moser, D. Spirn)

Ferromagnetic materials can be modeled by a vector field $\mathbf{m} : \Omega \to S^2$, where Ω is a domain representing the magnetic sample. A reasonable simplified model for thin films is to use $\Omega \subset \mathbb{R}^2$ and to consider the energy functional

$$E_\varepsilon(\mathbf{m}) = \int_\Omega e_\varepsilon(\mathbf{m}) = \int_\Omega \left(\frac{1}{2}|\nabla \mathbf{m}|^2 + \frac{m_3^2}{2\varepsilon^2} \right)$$

with a Dirichlet boundary condition. As $\varepsilon \to 0$ one observes the formation of vortices, very similar to the complex Ginzburg-Landau energy studied by Bethuel-Brezis-Hélein [1] and many other authors. A crucial difference is, however, that in our case, the vector field can point up or down in the vortex center instead of having a zero. While this makes no difference for the energy, it is important for dynamics.

The time evolution of $\mathbf{m}$ is governed by the Landau-Lifshitz-Gilbert equation,

$$\mathbf{m} \times \partial_t \mathbf{m} + \alpha \partial_t \mathbf{m} = \mathbf{f}_\varepsilon, \tag{1}$$

where $\mathbf{f}_\varepsilon$ is the tangential part of the energy gradient,

$$\mathbf{f}_\varepsilon = \Delta \mathbf{m} + |\nabla \mathbf{m}|^2 \mathbf{m} - \frac{1}{\varepsilon^2}(m_3 \mathbf{e}_3 - m_3^2 \mathbf{m}).$$

This equation is a hybrid between the purely precessional Schrödinger map flow and the harmonic map heat flow. It is analogous to the complex Ginzburg-Landau equation

$$(i + \alpha)\partial_t u = \Delta u + \frac{1}{\varepsilon^2}(1 - |u|^2)u \tag{2}$$

The motion of vortices for (2) was studied by Miot [8] in the whole plane and by our group [2] in bounded domains. The result is that the vortex centers move according to a system of ODEs. A crucial assumption is the well-preparedness of

the initial data, meaning that the energy is minimal for the given vortex positions. In [3] this result is proved for (1). In both cases a coercivity property of the renormalized energy is used to show strong convergence away from the vortices, which in turn ensures that well-preparedness stays true. To prove the motion law, one passes to the limit in evolution laws for quantities describing the vortices. The quantity $\frac{1}{|\log \varepsilon|} \langle \partial_t \mathbf{m}, \nabla \mathbf{m} \rangle$ appears in these evolution laws. As its limit was not known, in [8, 2, 3] this quantity is cancelled at the vortex core using a suitable combination of test functions.

A more powerful proof, developed in [4], contains an argument that the quantity $\frac{1}{|\log \varepsilon|} \langle \partial_t \mathbf{m}, \nabla \mathbf{m} \rangle$ actually converges, simplifying the structure and making it possible to generalize the proof to situations involving spin-current driven vortex motion. The convergence proof can be made quantitative in ε, at least for the Ginzburg-Landau case, and using an estimate in [6] allows one to study situations with unbounded number of vortices and to pass to the hydrodynamic limit [7].

For the complex Ginzburg-Landau equations, another approach is available that uses compactness results for the space-time Jacobian, see [9].

In a rather different approach [5], the strong convergence is proved using PDE methods inspired by the theory of partial regularity. While energy concentration may happen, it only affects the motion law when they lead to a flipping of the vortex core polarization. This proof requires less well-preparedness. The resulting motion law can be written as

$$\pi \dot{a} + \kappa \pi q(t) \dot{a}^{\perp} = -\nabla W(a),$$

where κ is 1 for the complex Ginzburg-Landau case and 2 in the Landau-Lifshitz-Gilbert equations. The function $q(t)$ only appears in the LLG case (otherwise it is identically 1) and describes the vortex core polarization. It is a piecewise constant function taking values in $1 + 2\mathbb{Z}$. A typical switching process would be a change from $+1$ to -1 or vice versa. While such spontaneous switching may look like an odd result, the possibility of easy switching is an important point for applications, and a long-term goal of our ongoing research is to understand how switching can be induced by electric currents or magnetic fields.

References

[1] F. Bethuel, H. Brezis, F. Hélein, *Ginzburg-Landau vortices*. Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston Inc., Boston, MA, 1994.

[2] M. Kurzke, C. Melcher, R. Moser, D. Spirn, *Dynamics for Ginzburg-Landau vortices under a mixed flow*, Indiana Univ. Math. J. **58** (2009), 2597–2622.

[3] M. Kurzke, C. Melcher, R. Moser, D. Spirn, *Ginzburg-Landau vortices driven by the Landau-Lifshitz-Gilbert equation*, Arch. Rat. Mech. Anal. **199** (2011), 843–888.

[4] M. Kurzke, C. Melcher, R. Moser, *Vortex motion for the Landau-Lifshitz-Gilbert equation with spin transfer torque*, SIAM J. Math. Anal. **43** (2011), 1099–1121

[5] M. Kurzke, C. Melcher, R. Moser, D. Spirn, preprint.

[6] M. Kurzke, D. Spirn, *Quantitative equipartition of the Ginzburg-Landau energy with applications*, to appear in Indiana Univ. Math. J.

[7] M. Kurzke, D. Spirn, *Vortex liquids and the Ginzburg-Landau equations*. Preprint, 2011.

[8] E. Miot, *Dynamics of vortices for the complex Ginzburg-Landau equation*, Anal. PDE **2** (2009), 159–186.
[9] Serfaty, S., and Tice, I. *Ginzburg-Landau vortex dynamics with pinning and strong applied currents*, Arch. Rat. Mech. Anal. **201** (2011), 413–464.

Renormalized energy for points in the plane

Etienne Sandier
(joint work with Sylvia Serfaty)

We report here on results contained in the forthcoming paper [SS3]. Consider the energy of n points $x_1,\dots,x_n$ in the plane

$$w_n(x_1,\dots,x_n) = -\sum_{i\neq j} \log|x_i - x_j| + n\sum_i V(x_i).$$

One may investigate minimizers of w_n, but also the probability measure on $(\mathbb{R}^2)^n$

$$Q_{n,\beta} = \frac{e^{-\beta w_n(x_1,\dots,x_n)}\, dx_1 \dots dx_n}{Z_{n,\beta}},$$

which arises in statistical mechanical models and random matrix models (for $V(x) = |x|^2$ and $\beta = 1, 2, 4$).

First Gamma limit. Minimizers were studied extensively (see for instance Saff-Totik [ST] and the references therein), especially in the classical case where V is zero on a given compact set, and $+\infty$ elsewhere, in which case the minimizers of w_n are known as Fekete points. In particular it is known under mild assumptions on V that, in modern language, $\frac{1}{n^2} w_n$ Gamma-converges to

$$I(\mu) = -\iint \log|x-y|\, d\mu(x)\, d\mu(y) + \int V\, d\mu,$$

where μ is any probability measure in $\mathbb{R}^2$. For the case of Fekete points this result is usually attributed to Fekete, Polya and Szegö ([F], [PZ]). Moreover, and this time the result goes back to Gauss, the unique minimizer μ_0 of I is characterized by the fact that for some constant c_0, the function

$$\zeta = -\log * \mu_0 + \frac{1}{2}V + c_0$$

is positive in $\mathbb{R}^2$ and vanishes on the support of μ_0. In the case $V(x) = |x|^2$, it is not difficult to deduce from this characterization that

$$\mu_0 = \frac{1}{\pi}\, dx \llcorner B(0,1).$$

This Gamma-convergence statement has a corresponding large deviations result, due tu Benarous and Zeïtouni [BZ] in the case $V(x) = |x|^2$: Large deviations for $Q_{n,\beta}$ holds with speed n^2 and rate function I.

Second Gamma limit. To further investigate properties of minimizing sequences for w_n, one possible route is to extract from them a limiting object more detailed than the limit μ of the empirical measures $\frac{1}{n}(\delta_{x_1}+\cdots+\delta_{x_n})$, from which one would then recover the term following $n^2 I(\mu)$ in the asymptotic expansion of $w_n(x_i)$ with respect to n. This object is akin to the Young measures on micropatterns introduced by Alberti-Müller [AM], and is defined as follows (we set $V(x)=|x|^2$ for convenience.)

Given $x_1,\dots,x_n$ we let H_n be the unique solution in $\mathbb{R}^2$ tending to 0 at $+\infty$ of

$$-\Delta H_n = 2\pi\left(\sum_{i=1}^n \delta_{\sqrt{n}x_i} - \frac{1}{\pi}\mathbf{1}_{B(0,\sqrt{n})}\right),$$

and

$$j_n = -\nabla^\perp H_n.$$

(The reason for considering j_n is that it will be a convergent quantity, whereas H_n may not be under our assumptions.) Then, fixing $p<2$ we have that $j_n \in L^p_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$, and that

$$\operatorname{curl} j_n = 2\pi\left(\sum_{i=1}^n \delta_{\sqrt{n}x_i} - \frac{1}{\pi}\mathbf{1}_{B(0,\sqrt{n})}\right), \quad \operatorname{div} j_n = 0.$$

We define the following probability measure on $L^p_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$

$$P_n = \fint_{B(0,\sqrt{n})} \delta_{j_n(x+\cdot)}\,dx.$$

Results.

Theorem 1. *The functionals*

$$F_n = \frac{1}{n}\left(w_n - n^2 I(\mu_0) - \frac{n}{2}\log n\right)$$

Gamma converge to

$$F(P) = \int W(j)\,dP(j),$$

where W is defined to be $+\infty$ outside the admissible class $\mathcal{A}\subset L^p_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ of vector fields satisfying $\operatorname{div} j = 0$ *and* $\operatorname{curl} j = 2\pi\left(\sum_{i=1}^\infty \delta_{x_i} - \frac{1}{\pi}\right)$ *with $\#\{i \mid |x_i|<R\} < CR$ for some constant independant of R. For $j\in\mathcal{A}$, $W(j) =$ " $\fint_{\mathbb{R}^2}|j|^2$ ".*

Here the quotation marks refer first to the fact that the average is over $\mathbb{R}^2$, hence needs to be defined more carefully, and second to the fact that if $j\in\mathcal{A}$ then $j\notin L^2$, which implies the need to take away the infinite part when computing the integral.

The above result yields as a byproduct, with some additional ingredients, several properties of $Q_{n,\beta}$. In particular, denoting by α the infimum of F, which is also the infimum of W on $\mathcal{A}$,

Theorem 2. *There exists a function $f(\beta)$ tending to 0 as $\beta \to +\infty$ such that*

$$\limsup_{n\to\infty} \frac{1}{n\beta} \log Q_{n,\beta}\left(F_n - \alpha > \delta\right) < -\delta + f(\beta).$$

Concluding remarks.

- The above results hold for potentials V wich are of the type $|x|^2$ in the sense that they must have as equilibrium measure a μ_0 which is supported on a smooth enough set of positive measure, with a density with respect to the Lebesgue measure which is smooth on this set. Note that when the density is not constant, the definition of P_n must be modified.
- Random matrix models on the real line, or the Fekete points case are under investigation.
- The Gamma convergence we prove is not exactly true as stated because the divergence free condition may be lost in the recovery sequence (see [SS3]).
- The minimization of the so-called renormalized energy W on $\mathcal{A}$ is an interesting problem in itself, which is thought to admit as a minimizer the triangular lattice. A partial result in this direction in [SS1] is that among simple lattices, the triangular lattice is minimizing.
- One of the main technical difficulties in proving the Gamma-convergence lies in the fact that F_n is the finite part of an L^2 norm, hence a positive quantity from which infinite quantities are subtracted. This causes problems in passing to the limit that are resolved by techniques initially devised for the analysis of Ginzburg-Landau vortices. (see [SS2]).

References

[PZ] Polya, G. and Szegö, G., *Über den transfiniten Durchmesser (Kapazitätskonstante) von ebenen räumlichen Punktmengen*, Jour. f. reine u. angewandte math. **165** (1931)

[F] Fekete, M., *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, math. zeits. **17** (1923).

[ST] E. Saff and V. Totik, *Logarithmic potentials with external fields*, Springer-Verlag, 1997.

[BZ] G. Ben Arous and O. Zeitouni, Large deviations from the circular law. *ESAIM Probab. Statist.* **2** (1998), 123–134.

[AM] G. Alberti and S. Müller, A new approach to variational problems with multiple scales. *Comm. Pure Appl. Math.* **54**, no. 7 (2001), 761-825.

[SS1] E. Sandier and S. Serfaty, From the Ginzburg-Landau model to vortex lattice problems, available at `http://arxiv.org/abs/1011.4617.`

[SS2] E. Sandier and S. Serfaty, Improved Lower Bounds for Ginzburg-Landau Energies via Mass Displacement, to appear in Analysis & PDEs, available at `http://arxiv.org/abs/1011.4616v1`.

[SS3] E. Sandier and S. Serfaty, 2D Coulomb Gases and the Renormalized Energy, in preparation.

Revisiting elasto-plasticity: the heterogeneous case

Gilles A. Francfort
(joint work with Alessandro Giacomini)

The first existence results in elasto-plasticity are due to P.-M. Suquet [5],[6], and C. Johnson [3]. This was completed by various works of R. Temam [7] and of R.V. Kohn and R. Temam [4]. That last paper focusses on the duality between stress fields (σ), or, more precisely, between the deviatoric part σ_D of σ, and plastic strains (p). The (time derivative of the) plastic strain p is merely a measure, while the stress field σ_D is typically not continuous, so that their product is not a priori meaningful. However, the analysis of the problem hinges on a good definition of that product which represents the mechanical dissipation.

Then, after a twenty five year mathematical lull, G. Dal Maso, A. De Simone and M.-G. Mora [2] revisited the existence of a quasi-static evolution as a time-parameterized set of minimization problems for the sum of the elastic energy and of the add-dissipation. The minimizing triplet(s) in displacement $u(t)$, elastic strain $e(t)$, and plastic strain $p(t)$ are kinematically compatible, that is that they should satisfy

$$Eu(t) := 1/2\left(Du(t) + (Du(t))^T\right) = e(t) + p(t)$$

at all times t. They should also be such that an energy conservation statement holds throughout the evolution. Once the existence of a variational evolution is established, it remains to show that the obtained time-dependent solution paths satisfy the more classical evolution laws for elasto-plasticity encountered in the mechanics literature. In particular, the flow rule states that, whenever the deviatoric stress $\sigma_D(t)$ (linearly related to $e(t)$ through Hooke's law) reaches the boundary of its admissible set K, the plastic strain $p(t)$ should flow in the direction normal to that set, that is $\dot{p}(t) \in \partial I_K(\sigma_D(t))$, where I_K stands for the indicatrix function of K. For this, the duality evoked earlier plays an essential role.

Here, we revisit that evolution, paying close attention to duality. We show that, roughly speaking, all needed results can be derived for a Lipschitz domain, and not only for C^2-domains, the required regularity if one follows [4]. Then, the heterogeneous case is investigated; the only manageable setting seems to be that of a domain Ω comprising a finite number of phases with piecewise C^2-interfaces, while the boundary of the domain is merely assumed to be Lipschitz. In particular, the general case of a set of admissible stresses $K(x)$ that would depend pointwise on the point x of the domain is open, unless the multi-function $x \multimap K(x)$ is piecewise continuous.

Since the plastic strain $p(t)$ is a bounded Radon measure – and its Lebesgue-singular part is the Lebesgue-singular part of the $BD(\Omega)$-function $u(t)$ – it can charge the interfaces, so that the dissipation functional must account for such an occurrence. The correct definition of that potential along the interfaces is as follows: at each point of the interface between two phases, say phases 1 and 2, and for each value of the plastic strain, the dissipation potential is the pointwise in space inf-convolution of that in either phase, but this for matrices of the form

$a \odot \nu$ only, where ν is the normal to the interface and $a \perp \nu$. By convex duality, this corresponds to choosing as admissibility set for the tangential part $(\sigma\nu)_\tau$ of the normal stresses $\sigma\nu$ to the interface the intersection of the admissibility sets $(K_1\nu)_\tau$ and $(K_2\nu)_\tau$, where K_i is the admissibility set for phase i. That condition is not akin to taking the intersection of the sets K_i on the boundary, which would produce the wrong dissipation.

The existence result is contingent upon a lower semi-continuity result for the dissipation which is tailored to the kinematic structure of elasto-plasticity and does not apply to any general sequence of weak-* converging Radon measures, unlike Reshetnyak's theorem. Then, using the extended duality that we introduced and the existence of the variational evolution, we recover the classical evolution, plus an interfacial flow rule which seems to be a missing ingredient in the literature on elasto-plasticity.

By the way, the same arguments also lead to a boundary flow rule which, to our knowledge, was only noted by G. Anzellotti ans S. Luckhaus [1], but seems to be equally absent from the mechanics literature. We contend that, absent such a flow rule, the classical elasto-plastic evolutions envisioned earlier, are underdetermined.

References

[1] G. Anzellotti and S. Luckhaus, *Dynamical Evolution of Elasto-Perfectly Plastic Bodies*, Appl Math Optim **15** (1987), 121–140.

[2] G. Dal Maso G., A. DeSimone A. and M.G. Mora M.G., *Quasi-static evolution problems for linearly elastic-perfectly plastic materials*, Arch. Ration. Mech. Anal. **180**-2 (2006), 237–291.

[3] C. Johnson, *Existence theorems for plasticity problems*, J. Math. Pures Appl. **55**-4 (1976, 431–444.

[4] R.V. Kohn and R. Temam, *Dual spaces of stresses and strains, with applications to Hencky plasticity*, Appl. Math. Optim. **10** (1983), 1–35.

[5] P.-M. Suquet, *Un espace fonctionnel pour les équations de la plasticité*, Ann. Fac. Sci. Toulouse Math. **5**-1 (1979), 77–87.

[6] P.-M. Suquet, *Sur les équations de la plasticité: existence et régularité des solutions*, J. Mécanique **20**-1 (1981), 3–39.

[7] R. Temam, Problèmes mathématiques en plasticité. Méthodes Mathématiques de l'Informatique, 12. Gauthier-Villars, Montrouge, 1983.

Inviscid limit of viscoelasticity with delamination

Tomáš Roubíček

Some mechanical systems exhibit discontinuous "catastrophic-like" response under mechanical loading; examples are damage, fracture, friction, or delamination (called also debonding). Stored energies driving such problems are necessarily *nonconvex*. A quasistatic formulation of such evolution processes is a well suited approach but one should rather carefully adopt a suitable concept of solution; cf. [5] for a menagerie of such concepts and their mutual comparison. We want to illustrate these features on a relatively simple problem of the Frémond-type [2]

delamination at small strains of the elastic body $\Omega \subset \mathbb{R}^d$ subjected to a time-dependent bulk load $f = f(t)$ which uses the nonconvex energy

$$\mathcal{E}(t,u,z) = \frac{1}{2}\int_\Omega \mathbb{C}e(u){:}e(u) - f(t){\cdot}u\,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_{\rm C}} z\mathbb{K}u{\cdot}u - \alpha(z)\,\mathrm{d}S$$

if the *delamination parameter* z ranges $[0,1]$ and if the displacement u satisfies the unilateral condition $u{\cdot}\vec{n} \le 0$ on the delaminating part $\Gamma_{\rm C}$ of $\partial\Omega$, otherwise $\mathcal{E}(t,u,z) = \infty$. If α is uniformly concave, we can speak about a *cohesive contact*.

The main concept well suited for mathematical analysis is the so-called energetic solutions, invented in [6, 7], based on global minimization of the recursive incremental problems $(u_\tau^k, z_\tau^k) \in \mathrm{Argmin}_{(u,z)}\mathcal{E}(k\tau,u,z) + \mathcal{R}(z{-}z_\tau^{k-1})$ with some phenomenological 1-homogeneous dissipation energy $\mathcal{R}$ reflecting the Griffith concept of an energy needed for (and dissipated by) delamination and with $\tau{>}0$ a time step. Especially in such nonconvex problems, the energetic solutions may jump sometimes unrealistically early, and thus some concepts based rather on a local minimization are more appropriate; when $\mathcal{E}(t,\cdot,z)$ is convex, this occurs typically due to domination of stored energy in a very large bulk like in [1, Sect.4.3]. This phenomenon was well recognized also in engineering literature, cf. e.g. [4].

A physically justified treatment of the phenomenon is by introducing a small viscosity into the evolution rule of internal parameters. Its asymptotics leads to a so-called vanishing-viscosity-solution concept. An alternative physically justified treatment is by introducing a viscous enough dissipative rheology into the equation for u; here Kelvin-Voigt model is to be considered.

Unless α is affine, rigorous treatment needs a modification by involving gradient theory for z to have it controlled in $W^{1,r}(\Gamma_{\rm C})$ with $r > d{-}1$. Better convergence also needs gradient theory for $e(u)$, the concept of so-called 2nd-grade non-simple materials involving the elasticity tensor $\mathbb{C}_2$. Denoting the relaxation time (i.e. viscosity in the Kelvin-Voigt model) by $\nu > 0$ and the solution (u_ν, z_ν) to the model $\mathrm{div}(\mathbb{C}e(u_\nu{+}\nu\frac{\partial u_\nu}{\partial t}){-}\mathrm{div}(\mathbb{C}_2\nabla e(u_\nu{+}\nu\frac{\partial u_\nu}{\partial t}))) = f$ with the boundary conditions characterized by a semi-stability $\mathcal{E}(t,u_\nu(t),z_\nu(t)) \le \mathcal{E}(t,u_\nu(t),\tilde z) + \mathcal{R}(\tilde z{-}z_\nu(t))$ for all admissible $\tilde z$'s and all times t's. This solution is continuous in time if the cohesive contact is considered, i.e. α is uniformly concave.

As inertia is neglected, we have uniform control of the BV-norm of u via the standard BV-estimate of z. This allows for a lot of convergence results if $\nu \to 0$. In particular, we have $u_\nu(t) \to u(t)$ in $H^2(\Omega;\mathbb{R}^d)$ and $z_\nu(t) \to z(t)$ in $W^{1,r}(\Gamma_{\rm C})$ weakly for all times t and strongly with exception only at most countable number of time instances t, and also $\mathcal{E}(t,u_\nu(t),z_\nu(t)) \to \mathcal{E}(t,u(t),z(t))$ with exception only at most countable number of time instances t's. The viscous-like dissipation energy $\nu e(\frac{\partial u_\nu}{\partial t}){:}e(\frac{\partial u_\nu}{\partial t}){+}\nu\nabla e(\frac{\partial u_\nu}{\partial t}){:}\nabla e(\frac{\partial u_\nu}{\partial t})$ converges weakly* to a 'defect-like' measure μ on $[0,\infty){\times}\bar\Omega$. In the limit, we thus get the energy balance

$$\mathcal{E}(t,u(t),z(t)) + \int_0^t\!\!\int_{\Gamma_{\rm C}} \mathcal{R}(\frac{\partial z}{\partial t})(\mathrm{d}S\mathrm{d}t) + \int_0^t\!\!\int_{\bar\Omega} \mu(\mathrm{d}x\mathrm{d}t) = E_0 + \int_0^t \mathcal{E}'_t(t,u(t),z(t))\,\mathrm{d}t$$

for a.a. $t \geq 0$ (with an inequality "$\leq$" for all t's) with $E_0 := \mathcal{E}(0, u(0), z(0))$. Completing it with equilibrium of the stresses on Ω and the semistability

$$\mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, u(t), \tilde{z}) + \mathcal{R}(\tilde{z} - z(t))$$

for all admissible $\tilde{z}$'s and all times t's, we obtain a definition of a certain *semi-energetic solution* (u, z, μ). This definition exhibits a concatenation property at least generically, i.e. for a.a. time intervals. Moreover, u and z are continuous in time except at most countable number of jumps.

It can be shown that, in a 1-dimensional experiment under increasing loading of a elastic bar with an adhesive contact, this 'semi-energetic' concept of solution leads to the same breakage time as the mentioned vanishing-viscosity solution, and always later than the energetic solution. Also 2-dimensional computations of a 'real' engineering problem clearly document the difference of the semi-energetic and the energetic concept.

After semi-implicit discretization, the Kelvin-Voigt viscous 'regularization' leads to recursive couples of strictly convex minimization problems, which enjoys the MPEEC (=Mathematical Programme with Evolution Equilibrium Constraints) structure and thus also gives a good basis for optimization of such system like in [3].

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References

[1] G. Bouchitté, A. Mielke, T. Roubíček, *A complete-damage problem at small strains*, Z. angew. Math. Phys. 60 (2009), 205–236.

[2] M. Frémond, *Dissipation dans l'adhrence des solides.*, C.R. Acad. Sci., Paris, Sér.II, 300 (1985), 709-714.

[3] M. Kočvara, M. Kružík, J.V. Outrata, *On the control of an evolutionary equilibrium in micromagnetics.* In: Optimization with Multivalued Mappings II (Eds.S.Dempe, V.Kalashnikov), Springer, 2006, pp.143–168.

[4] D. Leguillon, *Strength or toughness? A criterion for crack onset at a notch.* European J. of Mechanics A/Solids 21 (2002), 61–72

[5] A. Mielke, *Differential, energetic and metric formulations for rate-independent processes*, Lecture Series given at "C.I.M.E. Summer School on Nonlinear PDEs and Applications", Cetraro, June, 2008.

[6] A. Mielke, F. Theil, *On rate-independent hysteresis models*, Nonl. Diff. Eqns. Appl. 11 (2004), 151–189 (Accepted July 2001).

[7] A. Mielke, F. Theil, V.I. Levitas, *A variational formulation of rate–independent phase transformations using an extremum principle*, Arch. Rational Mech. Anal. 162 (2002), 137–177.

Elliptic regularization for gradient flows in metric spaces

Antonio Segatti
(joint work with R. Rossi, G. Savaré, U. Stefanelli)

The aim of this talk is to illustrate a novel variational view at gradient flows in metric spaces ([6],[7]). Given (X,d) a complete metric space, we consider the proper, lower semicontinuous, λ-geodesically convex and with compact sublevels functional

$$\phi : X \to (-\infty, +\infty].$$

λ-Geodesic convexity ($\lambda \in \mathbb{R}$) is the natural extension to metric spaces of the usual λ-convexity and means that every couple of points in $D(\phi) := \{\phi < \infty\}$ can be connected by a minimal and constant speed geodesic $\gamma : [0,1] \to X$ (thus satisfying $d(\gamma_s, \gamma_t) = (t-s)d(\gamma_0, \gamma_1)$), such that $\phi(\gamma_t) \leq (1-t)\phi(\gamma_0) + t\phi(\gamma_1) - \frac{\lambda}{2}t(1-t)d^2(\gamma_0, \gamma_1)$ for all $t \in [0,1]$.
The starting point is the minimization of the functionals $I^\varepsilon : AC^2(X) \to (-\infty, \infty]$ given, for $\varepsilon > 0$, by

$$I^\varepsilon(u) = \int_0^\infty \frac{\mathrm{e}^{-t/\varepsilon}}{\varepsilon} \left(\frac{\varepsilon}{2}|u'|^2(t) + \phi(u(t)) \right) \mathrm{d}t.$$

Here, $AC^2(X)$ is the set of absolutely continuous curves $t \in [0, +\infty) \mapsto u(t) \in X$, for which the *metric derivative* $t \mapsto |u'|(t) := \lim_{s \to t} d(u(t), u(s))/|t-s|$ exists a.e. and belongs to $L^2(0, +\infty)$ (see [1]). Now, for any $\varepsilon > 0$ and any $\bar{u} \in D(\phi)$, one can show that there exists $u_\varepsilon \in \operatorname{Arg\,min}_{v \in AC^2(X), v(0) = \bar{u}} I^\varepsilon(v)$. Then, we may ask the following question:
Does u_ε converges in some suitable sense to some curve u, with u a curve of maximal slope (for the functional ϕ, with respect to the upper gradient $|\partial\phi|$ and originating from $\bar{u}$)?
Curves of maximal slope turn out to be the natural extension to the metric setting of the concept of gradient flow and are defined as trajectories $u \in AC^2(X)$ such that $u(0) = \bar{u}$ and

$$\phi(u(t)) + \frac{1}{2}\int_0^t |u'|^2(t)\mathrm{d}t + \frac{1}{2}\int_0^t |\partial\phi|^2(u(t))\mathrm{d}t = \phi(\bar{u}) \quad \text{for all } t \geq 0. \tag{1}$$

The symbol $|\partial\phi|(u) := \limsup_{v \to u} (\phi(u) - \phi(v))^+ / d(u,v)$, for $u \in D(\phi)$, stands for the *local descending slope* of ϕ at u [1]. To see why the question of the convergence $u_\varepsilon \longrightarrow u$ is quite natural, let us briefly comment the Hilbert space scenario. If X is an Hilbert space, the minimizers u_ε are shown to satisfy the Euler Lagrange equation

$$-\varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + \partial\phi(u_\varepsilon(t)) \ni 0 \quad \text{for a.a. } t > 0,$$

thus, at least formally, when $\varepsilon \searrow 0$ one expects that u_ε approaches the solution of the gradient flow $u'(t) + \partial\phi(u(t)) \ni 0$. This argument has been made rigorous by Mielke & Stefanelli in [4]. Note that minimizing I^ε basically corresponds in addressing an *elliptic-in-time* regularization of the original gradient flow evolution. Our main result reads as follows (see [6] and [7]).

Theorem 1 (Variational principle). *As $\varepsilon \downarrow 0$, $u_\varepsilon \in \operatorname{Arg\,min}_{v \in AC^2(X), v(0)=\bar{u}} I^\varepsilon(v)$ admit a subsequence which locally uniformly converges to a curve of maximal slope.*

This convergence result entails the possibility of reformulating the differential problem (1) as a (limit of a class of) minimization problem(s). In particular, it paves the way to the application of the specific tools of the Calculus of Variations to (1), especially relaxation and Γ-convergence. As a by-product, we have an alternative existence proof for curves of maximal slope (see [1]). This variational approach has been firstly applied to rate-independent evolution by Mielke & Ortiz [3]. Then, two examples of relaxation of gradient flows via I^ε are shown in Conti & Ortiz [2] in the context of microstructure evolution. We point out that our interest in extending the above results to a purely metric setting is not at all academical, but rather motivated by applications to evolution PDEs with nonnegative solutions $u : \mathbb{R}^d \times [0, +\infty) \to [0, +\infty)$, in the form $\partial_t u - \nabla \cdot \big(u \nabla(\delta_u \phi(u))\big) = 0 \quad \text{in } \mathbb{R}^d \times [0, +\infty)$, where $\delta_u \phi(u)$ is the suitably defined first variation of an integral functional (even with nonlocal terms) as in [1] where the aforementioned PDE is reformulated as a gradient flow equation, in the metric space $\mathcal{P}_2(\mathbb{R}^d)$ of probability measures with finite second moment, endowed with the Wasserstein 2-metric (see also [5]). A key ingredient in the proof of Theorem 1 is the analysis of the *value function* $\bar{u} \in D(\phi) \mapsto V^\varepsilon(\bar{u})$ defined by $V^\varepsilon(\bar{u}) := \min_{u \in AC^2(X), u(0)=\bar{u}} I^\varepsilon(u)$. In particular, we show that V_ε satisfies some nice properties which resemble the Moreau-Yosida approximation of convex analysis. Moreover, V_ε is shown to satisfy a metric version of the *Dynamic Programming Principle*. Then we show that V_ε solves the corresponding (metric) Hamilton Jacobi equation which, eventually, implies the interesting fact that u_ε is a curve of maximal of maximal slope w.r.t. V_ε. Finally, relying on the λ-convexity of ϕ we obtain suitable a priori estimates on u_ε which allow to pass to the limit and conclude.

References

[1] L. Ambrosio, N. Gigli, G. Savaré, Gradient flows in metric spaces and in the space of probability measures. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2008. Second edition.

[2] S. Conti and M. Ortiz, *Minimum principles for the trajectories of systems governed by rate problems*, J. Mech. Phys. Solids, 56 (2008), 1885–1904.

[3] A. Mielke and M. Ortiz, *A class of minimum principles for characterizing the trajectories of dissipative systems*, ESAIM Control Optim. Calc. Var. 14 (2008), 494–516.

[4] A. Mielke and U. Stefanelli. *Weighted energy-dissipation functionals for gradient flows*, ESAIM Control Optim. Calc. Var., 17 (2011), 52-85.

[5] F. Otto. *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Partial Differential Equations, 26 (2001), 101–174.

[6] R. Rossi, G. Savaré, A. Segatti, U. Stefanelli, *A variational principle for gradient flows in metric spaces*, C. R. Acad. Sci. Paris Sér. I Math. **349** (2011), 1225–1228.

[7] R. Rossi, G. Savaré, A. Segatti, U. Stefanelli, *Weighted energy dissipation functionals for gradient flows in metric spaces*, in preparation (2011).

Least barriers to minimal hypersurfaces: an approach via MCF with obstacle

Emanuele Spadaro

In this talk I present some recent results on a mean curvature type equation for a geometric evolution with obstacle. The motivations come from a problem in differential geometry on the characterization of the least barrier for minimal hypersurfaces. More precisely, given a closed set $\Omega \subset \mathbb{R}^3$ with $\partial\Omega \in C^{1,1}$, which is the least set containing all minimal hypersurfaces with boundary in Ω?

A set $\Theta \subset \mathbb{R}^n$ is a *global barrier* if:

$$\Sigma \text{ minimal hypersurface, } \partial\Sigma \subset \Theta \quad \Longrightarrow \quad \Sigma \subset \Theta,$$

and we call *mean-convex hull* of Ω the least barrier containing Ω:

$$\Omega^{mc} := \bigcap_{\Omega \subset \Theta \in \mathcal{A}} \Theta, \tag{1}$$

where $\mathcal{A}$ denotes the family of global barriers in $\mathbb{R}^n$. (A similar notion of mean-convex hull have been introduced for minimal hypersurfaces spanning a fixed extreme boundary, see [3].)

Clearly, the closed convex hull Ω^{co} is a global barrier containing Ω, hence the intersection in (1) is non-trivial. Nevertheless, Ω^{co} may not be the smallest one. Moreover, if Ω^{mc} were smooth, it would turn out to be a mean-convex set. On the contrary, it is not difficult to see that a mean-convex set may fail to be a global barrier.

The result we show is the following.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$, with $n \leq 7$, be a bounded closed set with $\partial\Omega \in C^{1,1}$. Then, Ω^{mc} is a closed mean-convex set with $C^{1,1}$ regular boundary. Moreover, $\partial\Omega^{mc} \setminus \Omega$ is a minimal hypersurface with boundary in Ω.*

The proof of Theorem 1 goes through an evolution approach for a Mean Curvature Flow (MCF) with obstacle. The heuristic idea is to consider the evolution of the boundaries of sets F_t containing Ω such that the normal velocity $\vec{v}_{F_t}$ at any point of ∂F_t satisfies the equation:

$$\vec{v}_{F_t}(x) = \begin{cases} \vec{H}_{\partial F_t}(x) & \text{if } x \in \partial F_t \setminus \Omega, \\ \max\left\{ \vec{H}_{\partial F_t} \cdot \vec{n}_{F_t}, 0 \right\} \vec{n}_{F_t} & \text{if } x \in \partial F_t \cap \Omega, \end{cases} \tag{2}$$

where $\vec{n}_{F_t}$ denotes the unit external normal to ∂F_t. In words, the evolution of F_t follows the classical mean curvature flow equation away from the obstacle Ω while on the boundary of Ω satisfies a unilateral constraint, namely it can leave the obstacle if its mean curvature vector points outward, otherwise it stops. The idea is to show that

$$\Omega^{mc} = \lim_{t \to +\infty} F_t.$$

We suggest a variational formulation of (2) following the ideas in [1] and [2]. Let $E_0 \subseteq \mathbb{R}^n$ be the initial bounded closed set of the evolution such that

$$|E_0| = 0 \quad \text{and} \quad \Omega \subset E_0.$$

We define the approximate flow of time step $h > 0$ in the following way. We set $E_0^{(h)} := E_0$ and, given $E_i^{(h)}$ for some $i \in \mathbb{N}$, we let $E_{i+1}^{(h)}$ be a minimizer of the functional $\mathcal{F}(\cdot, h, E_i^{(h)})$ given by

$$\mathcal{F}(E, h, E_i^{(h)}) := \mathrm{Per}(E) + \int_{E \triangle E_i^{(h)}} \frac{\mathrm{dist}\big(x, \partial E_i^{(h)}\big)}{h}\, dx,$$

where $\mathrm{Per}(E)$ is the perimeter of a set E and the minimum is taken among all the sets E containing Ω a.e.,

$$\mathcal{F}(E_{i+i}^{(h)}, h, E_i^{(h)}) = \min\Big\{\mathcal{F}_{E_i^{(h)}}(E) \,:\, E \supset \Omega \ \text{ a.e.}\Big\}.$$

In particular, we consider the evolution of a minimizing hull E_0, i.e. a set satisfying

$$\mathrm{Per}(E_0) \le \mathrm{Per}(F) \quad \forall\, E \subseteq F \quad \text{such that} \quad F \setminus E \subset\subset \mathbb{R}^n.$$

It is possible to show that, starting from a minimizing hull a unique maximal flow $E_{\max,t}^{(h)}$ can be defined and the following holds.

Proposition 2. *For every $h > 0$ and $E_0 \subseteq \mathbb{R}^n$ minimizing hull with*

$$\Omega \subset E_0 \quad \textit{and} \quad |\partial E_0| = 0,$$

it holds:

(i) $E_{\max,t}^{(h)} \subseteq E_{\max,s}^{(h)}$ *for every* $0 \le s \le t$;

(ii) $E_{\max,t}^{(h)}$ *is a minimizing hull for every* $t \ge 0$;

(iii) $E_{\max,\infty}^{(h)} := \cap_{t\ge 0} E_{\max,t}^{(h)}$ *satisfies* $\partial E_{\max,\infty}^{(h)} \in C^{1,1}$ *with uniform estimates*

$$\|A_{\partial E_{\max,\infty}^{(h)}}\|_{L^\infty} \le c_0\, \|A_{\partial\Omega}\|_{L^\infty};$$

(iv) $\partial E_{\max,\infty}^{(h)} \setminus \Omega$ *is a smooth minimal hypersurface.*

The mean-convex hull Ω^{mc} is, finally, recovered as a double limit of the asymptotic sets $E_{\max,\infty}^{(h)}$:

$$\Omega_{mc} = \bigcap_{\varepsilon>0} \bigcup_{h>0} E_{\max,\infty}^{(h),\varepsilon},$$

where $E_{\max,\infty}^{(h),\varepsilon}$ are the asymptotic evolutions for the enhanced obstacles

$$\Omega_\varepsilon := \big\{x : \mathrm{dist}(x, \Omega) \le \varepsilon\big\}.$$

References

[1] F.J. Almgren, Jr., J.E. Taylor and L. Wang, *Curvature-driven flows: a variational approach*, SIAM J. Control Optim. 31 (2), 1993, pp.387–438.

[2] S. Luckhaus and T. Sturzenhecker, *Implicit time discretization for the mean curvature flow equation*, Calc. Var. Partial Differential Equations 3 (2), 1995, pp.253–271.

[3] B. White, *On the topological type of minimal submanifolds*, Topology 31 (2), 1992, pp.445–448.

Generalised functions of bounded deformation

Gianni Dal Maso

The space $GBD(\Omega)$ of *generalized functions of bounded deformation* on a bounded open set $\Omega \subset \mathbb{R}^n$ is defined by slicing. For every $\xi \in \mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ let $\Pi^\xi := \{y \in \mathbb{R}^n : y \cdot \xi = 0\}$ be the hyperplane orthogonal to ξ passing through the origin. For every set $B \subset \mathbb{R}^n$ and for every $y \in \Pi^\xi$ we define

$$B^\xi_y := \{t \in \mathbb{R} : y + t\xi \in B\}\,.$$

Moreover, for every $u\colon B \to \mathbb{R}^n$ we define the function $u^\xi_y\colon B^\xi_y \to \mathbb{R}$ by

$$u^\xi_y(t) := u(y + t\xi) \cdot \xi\,,$$

where the dot denotes the scalar product.

If $u\colon \Omega \to \mathbb{R}^n$ is $\mathcal{L}^n$-measurable, the *approximate jump set* J_u is the set of points $x \in \Omega$ for which there exist $u^+(x)$, $u^-(x) \in \mathbb{R}^n$, with $u^+(x) \neq u^-(x)$, and $\nu_u(x) \in \mathbb{S}^{n-1}$ such that

$$\operatorname*{ap\,lim}_{\substack{(y-x)\cdot\nu_u(x)>0\\ y\to x}} u(y) = u^+(x) \quad \text{and} \quad \operatorname*{ap\,lim}_{\substack{(y-x)\cdot\nu_u(x)<0\\ y\to x}} u(y) = u^-(x)$$

(we refer to [4] for the definition of the approximate limit).

For $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the jump set of u^ξ_y is denoted by $J_{u^\xi_y}$. Moreover we consider the set

$$J^1_{u^\xi_y} := \{t \in J_{u^\xi_y} : |(u^\xi_y)^+(t) - (u^\xi_y)^-(t)| \geq 1\}\,.$$

The space $GBD(\Omega)$ is defined as the space of all $\mathcal{L}^n$-measurable functions $u\colon \Omega \to \mathbb{R}^n$ such that there exists a bounded Radon measure λ on Ω with the following property: for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function u^ξ_y belongs to $BV_{loc}(\Omega^\xi_y)$ and

$$\int_{\Pi^\xi} \Big(|Du^\xi_y|(B^\xi_y \setminus J^1_{u^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{u^\xi_y}) \Big)\, d\mathcal{H}^{n-1}(y) \leq \lambda(B)\,,$$

for every Borel set $B \subset \Omega$, where D denotes the derivative in the sense of distributions. If we replace $BV_{loc}(\Omega^\xi_y)$ by $SBV_{loc}(\Omega^\xi_y)$ (see [2]), we obtain the definition of the space $GSBD(\Omega)$. The inclusions $BD(\Omega) \subset GBD(\Omega)$ and $SBD(\Omega) \subset GSBD(\Omega)$ follow from the results of [1].

We can prove two structure theorems for every $u \in GBD(\Omega)$ (see [3]). The first one concerns the approximate jump set J_u: this set is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable according to [4] and can be reconstructed from the jump sets of its one-dimensional slices u_y^ξ; more precisely, if $[u] := u^+ - u^-$ is the jump of u on J_u and $J_u^\xi := \{x \in J_u : [u](x) \cdot \xi \neq 0\}$, then $(J_u^\xi)_y^\xi = J_{u_y^\xi}$ for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. The second theorem concerns the *approximate symmetric gradient*: for $\mathcal{L}^n$-a.e. $x \in \Omega$ there exists a symmetric matrix, denoted by $\mathcal{E}u(x)$, such that

$$\operatorname*{ap\,lim}_{y \to x} \frac{\big(u(y) - u(x) - \mathcal{E}(x)(y-x)\big) \cdot (y-x)}{|y-x|^2} = 0\,.$$

We also prove the following analogue of the compact embedding of $BD(\Omega)$ into $L^1(\Omega;\mathbb{R}^n)$ (see [3]): every equi-integrable sequence u_k in $GBD(\Omega) \cap L^1(\Omega;\mathbb{R}^n)$ satisfying uniform bounds for the measures λ_k, considered in the definition, is relatively compact $L^1(\Omega;\mathbb{R}^n)$ and the limit of a subsequence belongs to $GBD(\Omega) \cap L^1(\Omega;\mathbb{R}^n)$.

From these results on $GBD(\Omega)$ we deduce the following compactness property for $GSBD(\Omega)$ (see [3]): if u_k is a sequence in $GSBD(\Omega)$ such that $\|u_k\|_{L^2(\Omega;\mathbb{R}^n)}$, $\|\mathcal{E}u_k\|_{L^2(\Omega;\mathbb{R}^{n\times n})}$, and $\mathcal{H}^{n-1}(J_{u_k})$ are bounded uniformly with respect to k, then there exist a subsequence, still denoted by u_k, and a function $u \in GSBD(\Omega)$, such that $u_k \to u$ in $L^1(\Omega;\mathbb{R}^n)$, $\mathcal{E}u_k \rightharpoonup \mathcal{E}u$ weakly in $L^1(\Omega;\mathbb{R}^{n\times n})$, and $\mathcal{H}^{n-1}(J_u) \leq \liminf_k \mathcal{H}^{n-1}(J_{u_k})$.

Thanks to this compactness result, $GSBD(\Omega)$ seems to provide the best functional framework for the study of variational models in linear elastic fracture mechanics, when the crack path is not prescribed.

References

[1] Ambrosio L., Coscia A., Dal Maso G.: Fine properties of functions with bounded deformation. *Arch. Ration. Mech. Anal.* **139** (1997), 201-238.

[2] Ambrosio L., Fusco N., Pallara D.: *Functions of bounded variation and free discontinuity problems*. Oxford University Press, Oxford, 2000.

[3] Dal Maso G.: Generalized functions with bounded deformation. Preprint SISSA, Trieste, 2011.

[4] Federer H.: *Geometric measure theory*. Springer-Verlag, Berlin, 1969.

Free energies, nonlinear flows and functional inequalities

Jean Dolbeault

(joint work with Giuseppe Toscani)

Consider on $\mathbb{R}^d$, $d \geq 3$, the fast diffusion equation

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \big(v\, \nabla v^{m-1}\big) = 0 \tag{1}$$

for some $m \in [m_1, 1)$ with $m_1 := (d-1)/d$. Assume that the initial data is a given nonnegative function u_0 in $L^1(\mathbb{R}^d)$ such that $u_0^m \in L^1(\mathbb{R}^d)$ and $|x|^2\, u_0 \in L^1(\mathbb{R}^d)$. Large time asymptotics of the solution are governed by Barenblatt self-similar

profiles, which can be studied either by comparison methods as in [7] or using time-dependent rescalings and free energy functionals. This second approach goes as follows.

Define the function u such that

$$v(\tau, y + x_0) = R^{-d}\, u(t,x)\;, \quad R = R(\tau)\;, \quad t = \tfrac12\,\log R\;, \quad x = \frac{y}{R} \tag{2}$$

where v is a solution of (1) with initial datum u_0. A simple computation shows that u has to be a solution of

$$\frac{\partial u}{\partial t} + \nabla\cdot\left[u\,\left(\sigma^{\frac{d}{2}(m-m_c)}\,\nabla u^{m-1} - 2\,x\right)\right] = 0 \quad t>0\;, \quad x\in\mathbb{R}^d\,, \tag{3}$$

with initial datum u_0 if we assume that R is chosen such that $R(0) = 1$. The diffusion coefficient σ given by

$$2\,\sigma^{-\frac{d}{2}(m-m_c)} = R^{\,1-d\,(1-m)}\,\frac{dR}{d\tau}\;.$$

A standard choice is to choose $\sigma = 1$ (which amounts to do a self-similar change of variables) and study the convergence of u as $t\to\infty$ towards the stationary solution B_1, where

$$B_\sigma(x) := \sigma^{-\frac{d}{2}}\,\left(C_M + \tfrac{1}{\sigma}\,|x|^2\right)^{\frac{1}{m-1}} \quad \forall\, x\in\mathbb{R}^d$$

and the constant C_M is chosen so that $\int_{\mathbb{R}^d} B_1\,dx = M := \int_{\mathbb{R}^d} u_0\,dx$. Consider the *free energy* and *relative Fisher information* functionals respectively defined by

$$\mathcal{F}_\sigma[u] := \frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m - B_\sigma^m - m\,B_\sigma^{m-1}\,(u-B_\sigma)\right]\,dx$$

$$\text{and}\quad \mathcal{J}_\sigma[u] := \sigma^{\frac{d}{2}(m-m_c)}\,\frac{m}{1-m}\int_{\mathbb{R}^d} u\,\left|\nabla u^{m-1} - \nabla B_\sigma^{m-1}\right|^2\,dx\;.$$

It has been established in [4] that

$$\mathcal{F}_\sigma[u] \le \frac14\,\mathcal{J}_\sigma[u]\;, \tag{4}$$

which amounts to an interpolation inequality of Gagliardo-Nirenberg type. As a special case for $m = m_1$, with $u = |f|^{2^*}$, it is equivalent to Sobolev's inequality

$$\|\nabla f\|_2^2 - \mathsf{S}_d\,\|f\|_{2^*}^2 \ge 0 \quad \forall\, f\in\mathcal{D}^{1,2}(\mathbb{R}^d) \tag{5}$$

where $2^* = \frac{2d}{d-2}$, S_d is the optimal constant of T. Aubin and G. Talenti and $\mathcal{D}^{1,2}(\mathbb{R}^d)$ is the completion with respect to the norm $\|\cdot\|$ defined by $\|f\|^2 = \|\nabla f\|_2^2 + \|f\|_{2d/(d-2)}^2$ of the set of smooth functions with compact support. Since

$$\frac{d}{dt}\mathcal{F}_\sigma[u(t,\cdot)] = -\mathcal{J}_\sigma[u(t,\cdot)] \tag{6}$$

if u is a solution of (3), we find that $\mathcal{F}_\sigma[u(t,\cdot)] \le \mathcal{F}_\sigma[u_0]\,e^{-4t}$ for any $t\ge0$, which proves the convergence of $u(t,\cdot)$ to B_1 in various norms if $\sigma = 1$. Such results have a nice interpretation in terms of gradient flows as was observed first in [8].

Rates of convergence are related to the following Hardy-Poincaré inequality. For any $\alpha \in (-\infty, \alpha_*) \cup (\alpha_*, 0)$, there is a positive constant $\Lambda_{\alpha,d}$ such that

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 \, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu_\alpha \quad \forall\, f \in L^2\left(d\mu_{\alpha-1}\right) \tag{7}$$

under the additional condition $\int_{\mathbb{R}^d} f \, d\mu_{\alpha-1} = 0$ if $\alpha < \alpha_*$. Here $\mu_\alpha(x) := (1+|x|^2)^\alpha$, has to be applied with $\alpha = 1/(m-1) < 0$ and $\alpha_* := -(d-2)/2$. The proof relies on the observation that $\mathcal{F}_\sigma[u(t,\cdot)]$ is asymptotically equivalent to $\int_{\mathbb{R}^d} |f|^2 \, d\mu_{\alpha-1}$ if $u(t,\cdot) = B_1\,(1 + f\, B_1^{1-m})$ while $\mathcal{F}_\sigma[u(t,\cdot)]$ can be controlled by $\int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu_\alpha$. The method covers a range which is actually not restricted to $m \in [m_1, 1)$: see [1] for details. Moreover, according to [2], for any $m \in (m_1, 1)$ there exists two constants $\mathsf{C} > 0$ and $\Lambda > 4$ such that

$$\mathcal{F}_\sigma[u(t,\cdot)] \le \mathsf{C}\, e^{-\Lambda t} \quad \forall\, t \ge 0 \tag{8}$$

if $x_0 = \frac{1}{M} \int_{\mathbb{R}^d} x\, u_0(x)\, dx$. The spectral gap given by (7) gives exactly $\Lambda = 4$ but the associated eigenspace is generated by the translations of B_1 and is discarded by the above choice of x_0. At this point no improvement is achieved in the Sobolev case, that is for $m = m_1$. The next eigenspace is generated by the dilations of B_1. It can also be discarded as it has been shown in [5], thus showing that Λ can be taken strictly larger than 4 even for $m = m_1$, but for the solution of a different equation. Namely, we shall consider the case where σ is now time-dependent and chosen in order to minimize $\sigma \mapsto \mathcal{F}_\sigma[u(t,\cdot)]$. As measured by the relative entropy, by doing so we are choosing the *best matching Barenblatt profile* $B_{\sigma(t)}$ among all possible ones. An easy computation shows that this amounts to fix $\sigma = \sigma(t)$ such that $\int_{\mathbb{R}^d} |x|^2 B_\sigma \, dx = \int_{\mathbb{R}^d} |x|^2 \, u(t,x) \, dx$, thus making (3) non-local, and (2) non explicit. Three main ingredients have now to be taken into account:

(i) Estimates (6) and (4) are unchanged in the new choice of $\sigma(t)$,

(ii) When applying the Bakry-Emery method, $\frac{d}{dt}\mathcal{J}_\sigma[u(t,\cdot)]$ involves an additional term which has the right sign because $\frac{d\sigma}{dt}$ can be related to $\mathcal{F}_\sigma[u(t,\cdot)]$,

(iii) Using $\|u\|_1 = \|B_\sigma\|_1$ and $\int_{\mathbb{R}^d} |x|^2 \, u \, dx = \int_{\mathbb{R}^d} |x|^2 \, B_\sigma \, dx$, one can prove a Csiszár-Kullback type inequality according to which we have

$$\frac{\mathcal{F}_\sigma[u]}{\sigma^{\frac{d}{2}(1-m)}} \ge \frac{m}{8 \int_{\mathbb{R}^d} B_1^m \, dx} \left(C_M \, \|u - B_\sigma\|_1 + \frac{1}{\sigma} \int_{\mathbb{R}^d} |x|^2 \, |u - B_\sigma| \, dx \right)^2 .$$

With $u = |f|^{2^*}$, we obtain an improvement of Sobolev's inequality (5), which gives an answer to the question of H. Brezis and E. Lieb in [3, Question (c), p. 75].

Theorem 1 ([6]). *Let $d \ge 3$. There is some explicit constant $\mathfrak{C}_d$ such that*

$$\|\nabla f\|_2^2 - \mathsf{S}_d \, \|f\|_{2^*}^2 \ge \frac{\mathfrak{C}_d}{\|f\|_{2^*}^{2\,\frac{3\,d+2}{d-2}}} \inf_{g \in \mathfrak{M}_d} \left\| |f|^{\frac{2d}{d-2}} - g^{\frac{2d}{d-2}} \right\|_1^4 \quad \forall\, f \in \mathcal{D}^{1,2}(\mathbb{R}^d) \, .$$

Here $\mathfrak{M}_d$ is the manifold of optimal functions for (5).

Estimate (8) is optimal in the large t regime, but the price we pay for it is that the constant C is not explicitly known in terms of the initial datum. On the

other hand, the result of Theorem 1 provides an improvement of the decay rate of $\mathcal{F}_\sigma[u(t,\cdot)]$ when it is large, that is for small values of t. This raises the open question of giving sharp estimates of the decay of $\mathcal{F}_\sigma[u(t,\cdot)]$ at any time $t \geq 0$.

References

[1] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J.-L. Vázquez. Asymptotics of the fast diffusion equation via entropy estimates. *Archive for Rational Mechanics and Analysis*, 191(2):347–385, 02 2009.

[2] M. Bonforte, J. Dolbeault, G. Grillo, and J. L. Vázquez. Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities. *Proceedings of the National Academy of Sciences*, 107(38):16459–16464, 2010.

[3] H. Brezis and E. H. Lieb. Sobolev inequalities with remainder terms. *J. Funct. Anal.*, 62(1):73–86, 1985.

[4] M. Del Pino and J. Dolbeault. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl. (9)*, 81(9):847–875, 2002.

[5] J. Dolbeault and G. Toscani. Fast diffusion equations: Matching large time asymptotics by relative entropy methods. *Kinetic and Related Models*, 4(3):701–716, 2011.

[6] J. Dolbeault and G. Toscani. Improved Sobolev's inequalities, relative entropy and fast diffusion equations. Submitted, 2011.

[7] A. Friedman and S. Kamin. The asymptotic behavior of gas in an n-dimensional porous medium. *Trans. Amer. Math. Soc.*, 262(2):551–563, 1980.

[8] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.

Kramers and non-Kramers Phase Transitions in Many-Particle Systems with Dynamical Constraint

Michael Herrmann

(joint work with Barbara Niethammer and Juan J.L. Velázquez)

We study the different dynamical regimes in a nonlocal Fokker-Planck equation and use formal asymptotics to derive reduced evolutionary models for different small-parameter limits.

Nonlocal Fokker-Planck equations were introduced in [2] to model many-particle storage systems such as lithium-ion batteries or interconnected rubber balloons. In the simplest case, see also [3], the equations read

$$\tau \partial_t \varrho(t,\, x) = \partial_x \Big(\nu^2 \partial_x \varrho(t,\, x) + \big(H'(x) - \sigma(t) \big) \varrho(t,\, x) \Big) \,,$$

$$\sigma(t) = \int_{\mathbb{R}} H'(x) \varrho(t,\, x) \, \mathrm{d}x + \tau \dot{\ell}(t) \,.$$

Here, ϱ is a time-dependent probability measure, $x \in \mathbb{R}$ denotes the state of a single particle, H is a generic double-well potential, and τ, ν are two parameters. Moreover, ℓ is a prescribed function of time that controls the first moment, that means we have

$$\int_{\mathbb{R}} x \varrho(t,\, x) \, \mathrm{d}x = \ell(t)$$

for any solution, provided that the initial data are admissible.

Numerical simulations as displayed in Figure 1, as well as heuristic arguments indicate that for $0 < \tau, \nu \ll 1$ there exist mainly two dynamical regimes.

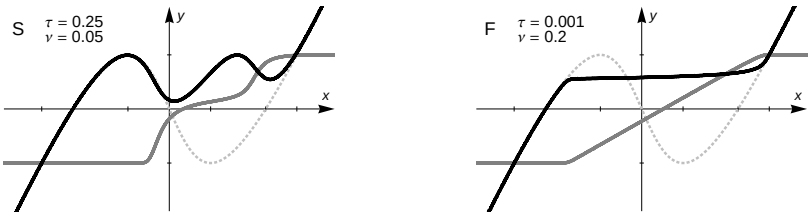

FIGURE 1. Typical solutions with $\dot{\ell} > 0$ for slow (left) and fast reactions (right). The solid curves in Black and Gray represent the evolution of σ and the phase fraction $\mu = \int_{\mathbb{R}} \operatorname{sgn}(x)\varrho \, \mathrm{d}x$, respectively.

The **fast reaction regime** corresponds to

$$\tau = \exp\left(-\frac{b}{\nu^2}\right), \qquad 0 < b < b_{\mathrm{crit}}, \qquad 0 < \nu \ll 1,$$

so phase transitions due to large deviations are possible. The main difficulty, however, is to understand how Kramers' formula [4] can be applied to an effective potential $H_\sigma(x) = H(x) - \sigma$ that depends implicitly on time t via the dynamical constraint ℓ.

As main result on fast reactions, we show the existence of two constants σ_b and d_b such that the limit dynamics for $\nu \to 0$ and $\dot{\ell} > 0$ is governed by

$$\sigma(t) = \begin{cases} \sigma_b & \text{for } t_1 < t < t_2, \\ H'\big(\ell(t)\big) & \text{else,} \end{cases} \qquad \dot{\mu}(t) = \begin{cases} d_b \, \dot{\ell}(t) & \text{for } t_1 < t < t_2, \\ 0 & \text{else.} \end{cases}$$

Here, the times $t_1 < t_2$ are uniquely defined by $H'\big(\ell(t_i)\big) = \sigma_b$ and $H''\big(\ell(t_i)\big) > 0$.

In the **slow reaction regime** we have

$$\nu = \exp\left(-\frac{a}{\tau}\right), \qquad 0 < a < a_{\mathrm{crit}}, \qquad 0 < \tau \ll 1,$$

and mass exchange according to Kramers' formula is not relevant anymore. Instead, the limit dynamics is governed by (i) quasi-stationary transport of either single-peak or two-peaks configurations, and (ii) a sequence of singular times corresponding to *switching*, *merging*, and *splitting* of peaks. We refer to Figure 2 for an illustration, and to [1] for more details.

REFERENCES

[1] M. Herrmann, B. Niethammer, J.J.L Velázquez, *Kramers and non-Kramers phase transitions in many-particle systems with dynamical constraint*, preprint arXiv:1110.3518 (2011).

[2] W. Dreyer, Cl. Guhlke, M. Herrmann, *Hysteresis and phase transition in many-particle storage systems*, Contin. Mech. Thermodyn. **23-3** (2011), 211–231.

[3] W. Dreyer, R. Huth, A. Mielke, J. Rehberg, M. Winkler, *Blow-up versus boundedness in a nonlocal and nonlinear Fokker-Planck equation*, WIAS-Preprint no. 1604 (2011).

[4] H.A. Kramers, *Brownian motion in a field of force and the diffusion model of chemical reactions*, Physica **7** (1940), 284–304.

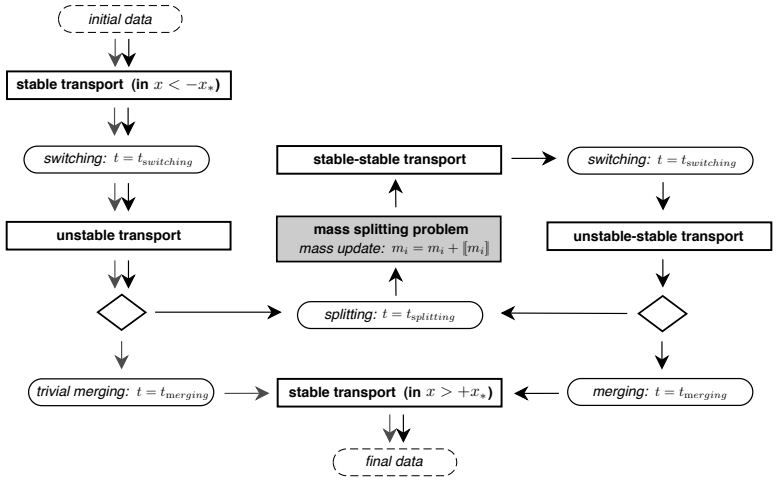

FIGURE 2. Flowchart of the limit dynamics with slow reactions, $\dot{\ell} > 0$, and $\ell(0) \ll 0$. Intervals of quasi-stationary transport are interrupted by several types of singular events.

Existence of Eulerian solution to the Semi-Geostrophic system on the 2D torus

GUIDO DE PHILIPPIS
(joint work with Luigi Ambrosio, Maria Colombo, Alessio Figalli)

The Semi-Geostrophic system. Consider the Semi Geostrophic system on the 2D torus

$$(SG) \qquad \begin{cases} \partial_t u_t^g(x) + \big(u_t(x)\cdot\nabla\big)u_t^g(x) + \nabla p_t(x) = -Ju_t(x) \\ u_t^g(x) = J\nabla p_t(x) \\ \nabla\cdot u_t(x) = 0 \end{cases}$$

with p_t and u_t $\mathbb{Z}^2-$periodic and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The stability principle of Cullen ad Purser (see [5, Chapter 3]) says that is natural to assume that $P_t = |x|^2/2 + p_t(x)$ is convex in $\mathbb{R}^2$. In particular to (SG) there exists an associated dual system (see [5] and [3, Appendix])

$$(SGD) \qquad \begin{cases} \partial_t \rho_t + \nabla\cdot(U_t\rho_t) = 0 \\ U_t(x) = J(x - \nabla P_t^*(x)) \\ \rho_t = (\nabla P_t)_\sharp \mathcal{L}_{\mathbb{T}^2} \\ P_0(x) = p^0(x) + |x|^2/2. \end{cases}$$

where P_t^* is the convex conjugate of P_t. Existence for (SGD) has been proved in 1998 by Benamou and Brenier, see [4]. However the existence of a distributional solution of (S) has remained open. In fact given a solution of (SGD) at a formal level a solution of SG is given by $(P_t - |x|^2/2, u_t)$ where

$$u_t = u_t(x) := [\partial_t \nabla P_t^*](\nabla P_t(x)) + [\nabla^2 P_t^*](\nabla P_t(x)) J(\nabla P_t(x) - x). \tag{1}$$

However a-priori P_t^* is just a convex function and its Hessian is just a Radon measure, due to this fact it is not clear which is the meaning to give to the previous equation. Recentely in [7] the authors succeed in proving $W^{2,1}$ regularity for solutions to the Monge-Ampère equation. As a consequence the second term appearing in the expression of the velocity field it is a well defined L^1 function; arguing as in [9] we can prove that also the first one belongs to L^1. The prove of the existence of distributional solutions now can be easily completed (see [3]).

Inspired by the theory of Regular Lagrangian Flows (see [1, 2]) we also show that the *weak Lagrangian solution* introduced in [6] are actually a RLF, in particular there exists a notion of flow associated to u_t.

Open Problems. In this section I report some open problems related to the study of (SG) and (SGD)

- Are distributional solution unique? This is unknown both for (SG) and for (SGD).
- Which is the relation between the uniqueness for (SG) and for (SGD)?
- Is the flow associated to u_t unique? (note that the folow associated to U_t is unique thatnks to the Theory developed in [1])
- If the initial data are smooth there exists a smooth solution? This has been shown in [10] for small times.

References

[1] L.Ambrosio: *Transport equation and Cauchy problem for BV vector fields.* Inventiones Mathematicae, **158** (2004), 227–260.

[2] L.Ambrosio: *Transport equation and Cauchy problem for non-smooth vector fields.* Lecture Notes in Mathematics "Calculus of Variations and Non-Linear Partial Differential Equations" (CIME Series, Cetraro, 2005) **1927**, B. Dacorogna, P. Marcellini eds., 2–41, 2008.

[3] L. Ambrosio, M. Colombo, G. De Philippis, A. Figalli , *Existence of Eulerian solutions to the semigeostrophic equations in physical space: the 2-dimensional periodic case.* Preprint 2011.

[4] J.-D.Benamou, Y.Brenier: *Weak existence for the semigeostrophic equation formulated as a coupled Monge-Ampère/transport problem.* SIAM J. Appl. Math., **58** (1998), 1450–1461.

[5] M.Cullen: *A mathematical theory of large-scale atmosphere/ocean flow.* Imperial College Press (2006).

[6] M.Cullen, M.Feldman: *Lagrangian solutions of semigeostrophic equations in physical space.* SIAM J. Math. Anal., **37** (2006), 1371–1395.

[7] G.De Philippis, A.Figalli: $W^{2,1}$ *regularity for solutions of the Monge-Ampère equation.* Preprint 2011.

[8] G.De Philippis, A.Figalli: *Second order stability for the Monge-Ampère equation and strong Sobolev convergence of optimal transport maps.* Preprint 2011.

[9] G.Loeper: *On the regularity of the polar factorization for time dependent maps.* Calc. Var. Partial Differential Equations, **22** (2005), 343–374.

[10] G.Loeper: *A fully non-linear version of the incompressible Euler equations: The semi-geostrophic system.* SIAM J. Math. Anal., **38** (2006), 795–823

Heat Flow on Finsler Spaces

Karl-Theodor Sturm
(joint work with Shin-Ichi Ohta)

A *Finsler manifold* is a smooth manifold M equipped with a *norm* $F_x(\cdot) : T_xM \to \mathbb{R}_+$ on each tangent space. More generally, $F_x(.)$ might be chosen to be a Minkowski norm (i.e. symmetry $F_x(v) = F_x(-v)$ is not required). Mostly, we will require that this norm is strongly convex and smooth and that it depends smoothly on the base point x. The particular case of a *Hilbert norm* on each tangent space leads to the important subclasses of Riemannian manifolds where the heat flow is widely studied and well understood. In a series of papers with Shin-ichi Ohta [OS1] we analyzed heat flow on a Finsler manifold. It can be defined

- either as gradient flow on $L^2(M, m)$ for the *energy*

$$\mathcal{E}(u) = \frac{1}{2}\int_M F^2(\nabla u)\, dm;$$

- or as gradient flow on the *reverse* L^2-Wasserstein space $\mathcal{P}_2(M)$ of probability measures on M for the *relative entropy*

$$\mathrm{Ent}(u) = \int_M u \log u\, dm.$$

Both approaches depend on the choice of a measure m on M and then lead to the same *nonlinear* evolution semigroup. We prove $\mathcal{C}^{1,\alpha}$-regularity for solutions to the (nonlinear) heat equation on the Finsler space (M, F, m). Typically, solutions to the heat equation will not be $\mathcal{C}^2$. Moreover, we derive integrated upper Gaussian estimates á la Davies and pointwise comparison results á la Cheeger-Yau [OS1].

The latter requires appropriate lower bounds on the curvature. It turned out that flag Ricci curvature is the appropriate notion. Indeed, this is exactly the quantity which is characterized by the curvature condition

$$\mathrm{Ent}(\mu_t) \le (1-t)\mathrm{Ent}(\mu_0) + t\mathrm{Ent}(\mu_1) - K\frac{t(1-t)}{2}W_2^2(\mu_0, \mu_1).$$

in the sense of Lott-Sturm-Villani. Surprisingly enough, however, this K-convexity of the relative entropy does not imply any bound of the form

$$W_2(p_t\mu, p_t\nu) \le e^{K't} W_2(\mu, \nu) \tag{1}$$

for the heat flow – which already was identified as the gradient flow of the relative entropy. Such an exponential growth bound (1) essentially holds true only in Riemannian cases. For general gradient flows of some functional S on a Finsler space (like the Wasserstein space on a given Finsler space), it is equivalent to the so-called skew-convexity of S, a property identified in [OS2]. In Riemannian

settings, convexity and skew-convexity coincide. In general, these are two entirely different properties.

In the linear theory, the L^1-version of property (1) is equivalent to the bound

$$\mathrm{Lip}(p_t f) \le e^{K' t}\mathrm{Lip}(f). \tag{2}$$

Despite the fact that (1) typically is not true, in [OS3] we could deduce a far reaching extension of (2), the so-called Bakry-Émery gradient estimate

$$\|\nabla p_t f\|^2(x) \le e^{-2Kt} q_t\big(\|\nabla f\|^2\big)(x).$$

(Here q_t denotes the semigroup for an appropriate linearization of the heat flow.) This is a consequence of the more fundamental Bochner inequality

$$\frac{1}{2}\Delta^{\nabla u}\big(\|\nabla u\|^2\big) - D(\Delta u)(\nabla u) \ge K\|\nabla u\|^2 + \frac{1}{N}\,(\Delta u)^2$$

which could be proven to hold pointwise on $M_u = \{x \in M \,|\, \nabla u \neq 0\}$ and in distributional sense on all of M.

Another important consequence of the latter is the differential Harnack inequality à la Li and Yau

$$\|\nabla(\log u)(t,x)\|^2 - \theta\partial_t(\log u)(t,x) \le N\theta^2\left(\frac{1}{2t} + \frac{K \wedge 0}{4(\theta-1)}\right)$$

for nonnegative solutions u to the heat equation on $(0,T] \times M$ or in integrated form

$$u(s,x) \le u(t,y)\cdot\left(\frac{t}{s}\right)^{\theta N/2} \exp\left(\frac{\theta d(x,y)^2}{4(t-s)} + \frac{\theta(K\wedge 0)N(t-s)}{4(\theta-1)}\right)$$

for any $\theta > 1$, $0 < s < t < T$ and $x, y \in M$.

References

[OS1] S. Ohta, K.-T. Sturm, *Heat flow on Finsler manifolds*. Comm. Pure. Appl. Math. **62** (2009), 1386–1433

[OS2] S. Ohta, K.-T. Sturm, *Non-contraction of heat flow on Minkowski spaces*. Sep 2010, to appear in Arch. Ration. Mech. Anal.

[OS3] S. Ohta, K.-T. Sturm, *Bochner-Weitzenböck formula and Li-Yau estimates on Finsler manifolds*. Apr 2011, http://arxiv.org/abs/1105.0983

Transition paths of maximal probability

Florian Theil

(joint work with Andrew Stuart, Frank Pinski)

Chemical reactions can be modeled via diffusion processes conditioned to make a transition between specified molecular configurations representing the state of the system before and after the chemical reaction. In particular the model of Brownian dynamics – gradient flow subject to additive noise – is frequently used. If the chemical reaction is specified to take place on a given time interval, then the

most likely path taken by the system is a minimizer of the Onsager-Machlup functional. The $\Gamma-$limit of this functional is determined explicitly in the case where the temperature is small and the transition time scales as the inverse temperature.

Consider the following conditioned SDE for $x \in C([0,T];\mathbb{R}^N)$ making a transition between two states x^- and x^+ in time T:

$$\begin{aligned} &\mathrm{d}x = -\nabla V(x)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W, \\ &x(0) = x^- \quad \text{and} \quad x(T) = x^+. \end{aligned} \tag{1}$$

We assume throughout that $V \in C^3(\mathbb{R}^N,\mathbb{R})$ and

(1) the set of critical points

$$\mathcal{E} = \{x \in \mathbb{R}^N \mid \nabla V(x) = 0\}$$

is finite;

(2) the Hessian $D^2V(x)$ has no zero eigenvalues for every $x \in \mathcal{E}$;

(3) the weak coercivity condition

$$\exists R > 0 \text{ such that } \inf_{|x|>R} |\nabla V(x)| > 0$$

is satisfied.

To enforce the scaling of interest we choose $T = \varepsilon^{-1}$ and rescale time as $t = \varepsilon^{-1}s$. Girsanov's formula, together with an integration by parts yields that the path density π_ε is proportional to $\exp\big(-\frac{1}{2\varepsilon}I_\varepsilon(x)\big)$ where the Onsager-Machlup functional $I_\varepsilon : H^1_\pm\big((0,1)\big) \to \mathbb{R}$ is defined by

$$I_\varepsilon(x) := \int_0^1 \Big(\frac{\varepsilon}{2}\left|\frac{\mathrm{d}x}{\mathrm{d}s}\right|^2 + \frac{1}{2\varepsilon}|\nabla V(x)|^2 - \varepsilon\Delta V(x)\Big)\mathrm{d}s. \tag{2}$$

Minimizers of the Onsager-Machlup functional have received considerable attention in the chemistry literature, e.g. [2]. To characterize the asymptotic behavior of minimizing sequences of I_ε we introduce the auxiliary functional

$$J(x) = \int_{-\infty}^{\infty} \Big(\frac{1}{2}\left|\frac{\mathrm{d}x}{\mathrm{d}s}\right|^2 + \frac{1}{2}|\nabla V(x)|^2\Big)\mathrm{d}s,$$

and define the transition energy

$$\Phi(x^-,x^+) = \inf\left\{J(y) \;\middle|\; y \in X(x^-,x^+)\right\},$$

where the set of admissible paths is defined as

$$X(x^-,x^+) = \left\{y \in BV(\mathbb{R}) \;\middle|\; \lim_{t\to\pm\infty} y(t) = x^\pm \text{ and } \dot{y} \in L^2(\mathbb{R})\right\}.$$

The existence of minimizer of J in $X(x^-,x^+)$ cannot be expected without additional assumptions on the initial state x^- and the final state x^+. However, if we allow for intermediate transition states we can establish a direct representation of the transition energy Φ which involves only minima and avoids the usage of infima.

Proposition 1. *Let V be admissible and $x^{\pm} \in \mathcal{E}$ be two critical points of V. Then there exists a finite sequence $\{x_i\}_{i=0}^{k} \in \mathcal{E}$ such that $x_0 = x^-$, $x_k = x^+$ and*

$$\Phi(x^-, x^+) = \sum_{i=1}^{k} \min \{ J(y) \mid y \in X(x_{i-1}, x_i) \}.$$

We observe that $X(x^-, x^+)$ can be computed explicitly in many situations. This result establishes a clear link to transition state theory although the setup prevents a direct comparison.

Proposition 2. *If $x^\star \in X(x^-, x^+)$ is a minimizer of J and either x^- or x^+ is a local extremum of V then $x^\star$ solves*

$$\frac{\mathrm{d}x^\star}{\mathrm{d}s} = \pm \nabla V(x^\star) \tag{3}$$

and

$$\Phi(x^-, x^+) = \big| V\big(x^+\big) - V\big(x^-\big) \big|. \tag{4}$$

A compact representation of the results is available in the form of the Γ-limit of I_ε.

Theorem 3. *The Γ-limit of the functional I_ε as ε tends to 0 is*

$$I_0(x) = \begin{cases} \sum\limits_{\tau \in \mathcal{D}} \Phi(x^-(\tau), x^+(\tau)) - \int_0^1 \Delta V(x(s))\, \mathrm{d}s & \text{if } x \in BV([0,1]) \\ & \text{and } x(s) \in \mathcal{E} \text{ a.e. } s \in [0,1], \\ & \\ +\infty & \text{else,} \end{cases}$$

where $\mathcal{D}(x)$ is the set of discontinuity points of x and $x^{\pm}(\tau)$ are the left and right-sided limits of x at τ.

References

[1] D. Dürr & A. Bach. The Onsager-Machlup function as lagrangian for the most probable path of a diffusion process. *Communications in Mathematical Physics* **160** 153–170, 1978.

[2] R. Olender and R. Elber. Yet another look at the steepest descent path. *J. Mol. Struc: THEOCHEM* **63** 398–399, 1997.

Analysis of Lennard-Jones interactions in 2D

Andrea Braides

(joint work with M.S. Gelli)

The behaviour of systems of Lennard-Jones atomic interactions as the number of atoms N goes to $+\infty$ has been thoroughly analyzed in one dimension. After introducing an increasing parameterization the position of each atom is interpreted as a displacement from a regular configuration, and we obtain a limit continuum theory of brittle fracture with an internal parameter and opening fracture (see [10, 3, 4, 6]). Here, we consider a two-dimensional system of nearest-neighbour

Lennard-Jones interactions, for which we have to make some simplifying assumptions. A first one is to consider displacements as perturbation from a ground state for which *crystallization* (see [9]) holds on the triangular lattice $\mathbb{T}$ (or rather on a bounded portion Λ of $\mathbb{T}$). For such perturbations it makes sense to assume that only short-range interactions be taken into account.

In order to prevent undesired "foldings" at a discrete level without considering longer-range interactions, Friesecke and Theil [8] proposed to add a three-point condition on neighbouring nodes. In the case of a triangular lattice, this condition simply amounts to requiring that

$$\det \nabla u > 0,$$

where u is the affine interpolation of the function defined on the vertices of each triangle. We use the terminology and techniques of Γ-convergence applied to a discrete-to-continuum analysis [2]; in this framework we study the overall behaviour of the energies as the size of Λ diverges, by considering $\Lambda = \frac{1}{\varepsilon}\Omega \cap \mathbb{T}$, with Ω a fixed bounded domain in $\mathbb{R}^2$, and using $\Omega \cap \varepsilon\mathbb{T}$ as the set of parameters. We focus on the *surface scaling*; i.e., the energies we are going to examine will be of the form

$$F_\varepsilon(u) = \sum_{i,j \in \Omega \cap \varepsilon\mathbb{T}, |i-j|=\varepsilon} \varepsilon \Big(J\Big(\Big\| \frac{u_i - u_j}{\varepsilon} \Big\| \Big) - \min J \Big)$$

(we have scaled the energies by an additive constant so that the energy density is always positive), where u_i is the value of the discrete function u at the node $\varepsilon i \in \Omega \cap \varepsilon\mathbb{T}$, and the piecewise-affine interpolation of u on the triangulation related to $\varepsilon\mathbb{T}$ is supposed to satisfy the positive-determinant constraint. Under these assumption we address the two issues

- determine whether some condition of "opening crack" still hold in the two-dimensional case;
- characterize a limit surface energy defined on functions defined on Ω.

Other two issues present in the one-dimensional analysis; i.e., the characterization of the bulk energy close to ground states and surface relaxation have been separately addressed by Braides, Solci and Vitali (for the bulk analysis) [5] and Theil [9] (for the external surface relaxation).

We note that by [7] gradients of limits of sequences (u_ε) with equi-bounded energy are piecewise rotations with an underlying partition of Ω into sets of finite perimeter. On the boundaries of such sets we have a normal ν on whose two sides we have the values $u^\pm(x)$ of u and two rotations $R^\pm$ among those labeling the sets of the partition. Note that interfaces are limits of triangles of side-length ε which are deformed by u_ε into triangles with one side (actually two) of diverging length but with the same ordering of the vertices. If only one layer of triangles is deformed that gives a limit interface, then this produces a relation between ν, $u^\pm(x)$ and $R^\pm$. In the simplest case when ν is orthogonal to one of the unit vectors of $\mathbb{T}$ (we call these *coordinate normals*), this relation reads as

$$\langle u^+(x) - u^-(x), R^\pm \nu \rangle \geq 0\,,$$

and can be regarded as an *opening-crack condition* in the "finite" case. In such case the surface density can be computed and gives a crystalline norm with hexagonal symmetries as for spin systems in the same geometry (see [1]).

If ν is not a coordinate normal then the opening-crack condition is more complicated, due to microscopic anisotropies of the lattice, which disfavor cracks not orthogonal to lattice directions. However, the situation described above is not the only possibility, since more than one layer of triangles may be "strongly deformed". This gives a higher energy on the interface, but relaxes the constraints on ν, $u^{\pm}(x)$ and $R^{\pm}$. Moreover, additional energy contributions may be given by points where three or more interfaces meet; in this case, even though the opening-crack condition above may be satisfied on each interface, the system of interfaces may be incompatible with the positive-determinant condition in their common point. It must be observed that by introducing a high number of extra interfaces at the discrete level, the finite opening-crack condition on interfaces and the positive-determinant constraint at meeting points can be removed altogether, at the expenses of a complex non-local form of the final energy.

References

[1] R. Alicandro, A. Braides and M. Cicalese, *Phase and anti-phase boundaries in binary discrete systems: a variational viewpoint*, Netw. Heterog. Media **1** (2006), 85–107.

[2] A. Braides, Γ-*convergence for Beginners*, Oxford University Press, Oxford, 2002.

[3] A Braides, G. Dal Maso and A. Garroni, *Variational formulation of softening phenomena in fracture mechanics: the one-dimensional case*, Arch. Rational Mech. Anal. **146** (1999), 23–58.

[4] A. Braides, A.J. Lew and M. Ortiz, *Effective cohesive behaviour of layers of interatomic planes*, Arch. Ration. Mech. Anal. **180** (2006), 151–182.

[5] A. Braides, M. Solci, and E. Vitali, *A derivation of linear elastic energies from pair-interaction atomistic systems*, Netw. Heterog. Media **2** (2007), 551–567.

[6] A. Braides and L. Truskinovsky, *Asymptotic expansions by Gamma-convergence*, Cont. Mech. Therm. **20** (2008), 21–62.

[7] A. Chambolle, A. Giacomini and M. Ponsiglione, *Piecewise rigidity*, J. Funct. Anal. **244** (2007) 134–153.

[8] G. Friesecke and F. Theil, *Validity and failure of the Cauch-Born rule*, J. Nonlin. Sci. **12** (2002), 445–478.

[9] F. Theil, *A proof of crystallization in two dimensions*, Comm. Math. Phys. **262** (2005), 209–236.

[10] L. Truskinovsky, *Fracture as a phase transition*, Contemporary research in the Mechanics and Mathematics of materials (Batra and Beatty eds.) CIMNE, Barcelona, 1996, 322–332.

Singularity at the tip of a 2D fracture

Antonin Chambolle

(joint work with G. Francfort, A. Lemenant, J.-J. Marigo)

The Griffith theory, in planar elasticity, assumes that the elastic energy wich is dissipated for opening a crack in a linear elastic material is proportional to the length of the crack. It is also based on the assumption that the crack will grow continuously, along a given path. Many attempts have been done to understand

how to predict the correct crack path. We address particularly in this talk the issue of "kinking", that is, the brutal change of direction of a fracture which is loaded in a non-symmetric way. First, we have recalled our previous work on the subject in collaboration with G. A. Francfort and J.-J. Marigo [6, 7].

In that work, we have considered the global energy+dissipation functional

$$\mathcal{E}(u,\Gamma) \;=\; \int_{\Omega\setminus\Gamma} Ce(u) : e(u)\,dx \;+\; \mathcal{H}^1(\Gamma), \tag{1}$$

defined for a fracture Γ and a displacement $u \in H^1_{loc}(\Omega \setminus \Gamma)$ satisfying a boundary condition $u = u^0$ on $\partial\Omega$.

Then, we have considered the different criteria for deciding the direction in which a crack should grow, at the tip of a straight initial fracture Γ^i. It is known since [15] that at the tip (assumed to be at the origin), the minimal u (which we also denote by u^0) is singular, with a singularity of the form $K_I\phi_I + K_{II}\phi_{II}$, where ϕ_x are 1/2-homogeneous displacements, the I eigenmode corresponding to a pure opening (symmetric w/r the crack) and the II eigenmode corresponding to a shear opening, where the lips of the fracture move tangentially. If $K_{II} = 0$ the fracture is expected to grow in a straight direction, as soon as its *energy relase rate* is large enough.

If $K_{II} \neq 0$, the is expected to turn, and it raises stability issues. In [6, 7], we have considered a new (in)stability criterion: we claim that the fracture should be unstable as soon as at any small scale $\epsilon > 0$, there exists an add-crack Γ (we assume it is a connected, compact one-dimensional set containing the tip) of length ϵ such that the release of energy

$$\frac{1}{\epsilon}\mathcal{F}(\Gamma) \;=\; -\frac{1}{\epsilon}\left(\min_{u\in H^1(\Omega\setminus(\Gamma^i\cup\Gamma))}\int_{\Omega\setminus(\Gamma^i\cup\Gamma)} Ce(u) : e(u)\,dx - \int_{\Omega\setminus\Gamma^i} Ce(u^0) : e(u^0)\,dx\right) \tag{2}$$

is larger than 1. This criterion must be satisfied for crack evolutions built by incremental minimizations of (1), such as proposed in [11, 10, 5, 13, 17] (the last of which involves local minimizations).

More precisely, letting (for Γ of length ~ 1) $\mathcal{F}_\epsilon(\Gamma) = \frac{1}{\epsilon}\mathcal{F}(\epsilon\Gamma)$, we show that if $\Gamma_\epsilon \to \Gamma$ is the Hausdorff sense as $\epsilon \to 0$ then $\mathcal{F}_\epsilon(\Gamma) \to \mathcal{F}_0(\Gamma)$, where the expression $\mathcal{F}_0$ is given in terms of a minimization problem in $\mathbb{R}^2 \setminus (\mathbb{R}_- \times \{0\} \cup \Gamma)$, and which involves only the singular (1/2-homogeneous) part at the tip of u^0 (hence, essentially, is linear in K_I, K_{II}). We can show that there exists a (non straight) pattern $\bar\Gamma$ such that $\mathcal{F}(\bar\Gamma)$ is larger than any energy release rate predicted by the classical theory, which assumes straight (or smoothly growing) add-cracks. This shows that in theory, the system should be unstable for smaller loads u^0 than predicted by the classical theory.

The main issues in this study is that the non-interpenetrability of the material along the crack is never taken into account (the work [16] tries to deal with this issues, but it involves only straight, possibly kinking, cracks), and that we need to assume that the crack is flat near the tip (this could be released a little, but still

strong regularity needs to be assumed. A recent work [18] considers $C^{1,1}$ cracks, but only in the scalar setting).

With A. Lemenant, we are trying to extend these results to "arbitrary" cracks. In the scalar setting, we could show the following [8] result. We assume that $u \in H^1_{loc}(\Omega \setminus \Gamma^i)$ (with $\nabla u \in L^2$) solves

$$\begin{cases} \lambda u - \mathrm{div} A\nabla u = f & \text{in } \Omega \setminus \Gamma^i \\ A\nabla u \cdot \nu = 0 & \text{on both sides of } \Gamma^i \\ u = u_0 \in L^\infty(\partial\Omega) & \text{on } \partial\Omega \end{cases}$$

and that either $\lambda > 0$, $f \in L^\infty(\Omega)$, or $\lambda = 0$ and $f = 0$. We also assume that $A(x)$ is a Hölder-continuous positive definite tensor. Then it holds the theorem:

Theorem [8]. *Assume Γ^i is connected and has density $1/2$ at 0:*

$$\lim_{\rho \downarrow 0} \frac{\mathcal{H}^1(\Gamma^i \cap B_\rho)}{2\rho} \;=\; \frac{1}{2}$$

Then the limit

$$K^2 = \lim_{\rho \downarrow 0} \frac{1}{\rho} \int_{B_\rho \setminus \Gamma^i} A\nabla u \cdot \nabla u \, dx$$

exists and is finite. Moreover, up to rotations (assuming $A(0) = Id$), the blowups $u_\epsilon(y) = (u(\epsilon y) - u(0))/\sqrt{\epsilon}$ converge to

$$u_0^0 \;=\; K\sqrt{\frac{2r}{\pi}} \sin\frac{\theta}{2}.$$

The existence of the limit is based on simple generalizations of Bonnet's monotonicity formula [3]. The fact that the limit is a "crack-tip" function, with no loss of energy, is based on the adaption of standard result on the limit of elliptic problems in 2D sets with connected boundary [4, 5].

References

[1] M. Amestoy. *Propagation de fissures en élasticité plane.* Thèse de Doctorat d'Etat, Université Pierre et Marie Curie, Paris (1987).

[2] M. Amestoy and J.-B. Leblond. *Crack paths in plane situations. II. Detailed form of the expansion of the stress intensity factors.* Internat. J. Solids Structures, **29** (1992), no. 4, 465–501.

[3] A. Bonnet. *On the regularity of the edge set of Mumford-Shah minimizers.* In *Variational methods for discontinuous structures (Como, 1994)*, volume 25 of *Progr. Nonlinear Differential Equations Appl.*, Birkhäuser, Basel (1996), 93–103.

[4] D. Bucur and N. Varchon. *Boundary variation for a Neumann problem.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **29** (2000), no. 4, 807–821.

[5] A. Chambolle. *A density result in two-dimensional linearized elasticity, and applications.* Arch. Ration. Mech. Anal., **167** (2003), no. 3, 211–233.

[6] A. Chambolle, G. A. Francfort, and J.-J. Marigo. *When and how do cracks propagate?* J. Mech. Phys. Solids, **57** (2009), no. 9, 1614–1622.

[7] A. Chambolle, G. A. Francfort, and J.-J. Marigo. *Revisiting energy release rates in brittle fracture.* J. Nonlinear Sci., **20** (2010), no. 4, 395–424.

[8] A. Chambolle and A. Lemenant. *The Stress-Intensity Factor for nonsmooth fractures in antiplane elasticity.* Preprint (2011), oai:hal.archives-ouvertes.fr:hal-00634460
[9] B. Cotterell and J. R. Rice. *Slightly curved or kinked cracks.* International Journal of Fracture, **16** (1980), no. 2 155–169.
[10] G. Dal Maso, G. A. Francfort, and R. Toader. *Quasistatic crack growth in nonlinear elasticity.* Arch. Ration. Mech. Anal., **176** (2002), no. 2, 165–225.
[11] G. Dal Maso and R. Toader. *A model for the quasi-static growth of brittle fractures: existence and approximation results.* Arch. Ration. Mech. Anal., **162** (2002), no. 2, 101–135.
[12] G. A. Francfort and J.-J. Marigo. *Revisiting brittle fracture as an energy minimization problem.* J. Mech. Phys. Solids, **46** (1998), no. 8, 1319–1342.
[13] G. A. Francfort and C. J. Larsen. *Existence and convergence for quasi-static evolution in brittle fracture.* Comm. Pure Appl. Math., **56** (2003),, no. 10, 1465–1500.
[14] R.V. Goldstein and R.L. Salganik. *Brittle fracture of solids with arbitrary cracks.* Internat. J. Fracture, **10** (1974), 507–523.
[15] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA (1985).
[16] A. Tani, A.M. Khludnev, V.A. Kovtunenko. *On the topological derivative due to kink of a crack with non-penetration. anti-plane model.* J. Math. Pures Appl. (9) **94** (2010), no. 6, 571–596.
[17] C.J. Larsen *Epsilon-stable quasi-static brittle fracture evolution.* Commun. Pure Appl. Math. **63** (2010), 630–654.
[18] G. Lazzaroni and R. Toader. *Energy release rate and stress intensity factor in antiplane elasticity.* J. Math. Pures Appl. (9), **95** (2011), no. 6, 565–584.
[19] J.-B. Leblond. *Crack paths in plane situation – I, General form of the expansion of the stress intensity factors.* Int. J. Solids Struct., **25** (1989), 1311–1325.

Brittle Crack Propagation in Mixed Mode

Matteo Negri

Brittle crack propagation is currently a very active field of research in applied analysis. In the collection [1] of open problems in the mathematical theory of elasticity J. Ball suggested to *"clarify the status of models based on the energy functional F with respect to classical fracture mechanics and to non-linear elastostatics"*, where F stands for the energy functional

$$F(u) = \int_\Omega W^e(\varepsilon)\, dx + G_c\, \mathcal{H}^1(J_u)\,,$$

to be defined in SBV, SBD or the most recent $GSBD$ (depending on the setting). This talk presents indeed an analysis and some partial results on the classical criteria for the prediction of the path along which a crack evolves quasi-statically subject to a mixed mode loading. The talk focuses on regularity issues: on the crack and on the displacement field. A very recent result on the energy F can be found instead in the talk of G. Dal Maso.

We consider a two dimensional setting in which the reference configuration is an open, bounded Lipschitz set Ω, cut by a pre-existing straight crack Γ_0. Proportional Dirichlet boundary conditions are imposed on a portion $\partial_D\Omega$ of the boundary. When the setting is not symmetric (as it is in general) the crack does not propagate along the straight extension of the initial cut Γ_0, it extends instead

along a curved path with a sharp deflection angle (the kink) at initiation. Several theories have been advanced in Fracture Mechanics to explain, and to predict, the path (and the kink angle); among them, the most popular are probably the Principle of Local Symmetry (PLS) [4] and the Maximal Energy Release Rate (GMAX) [3].

To understand, and even state, these criteria it is first necessary to introduce the stress intensity factors. After [5] we know that for a straight crack, say Γ_0, the equilibrium configuration (for linearized isotropic elasticity) can be represented in the form

$$u_0 = K_I \rho^{1/2} U_I(\theta) + K_{II} \rho^{1/2} U_{II}(\theta) + \bar{u}\,,$$

for $\bar{u} \in H^2(B_r \setminus \Gamma_0)$. On the base of [6] (where actually the anti-plane setting is adopted) we can expect the above representation to hold for cracks of class $C^{1,1}$ or $C^{1,1}_{loc}$. The Stress Intensity Factors K_I and K_{II} are related respectively to Mode I (opening) and Mode II (sliding) on the faces of the crack Γ_0. When both are non-zero the crack undergoes a mixed mode loading.

Now, let us go back to the criteria: PLS and GMAX. The former states that the crack propagates only in Mode I (opening), i.e. along a path Γ_s that satisfies $K_{II}(\Gamma_s) = 0$. The latter states that the crack propagates following the direction of steepest descent of the elastic energy, i.e. the direction v that makes the Energy Release Rate G_v maximal. Note that G_v as well relies on the SIF since it is given by a quadratic form for K_I and K_{II}. Both the criteria are theoretically sound and consistent with the experimental observations. For a comparative analysis of PLS and GMAX we refer to [2].

We employ a system of coordinates with the origin at the crack tip and with $\hat{e}_1$ aligned with Γ_0. The crack path will be represented by the graph of a Lipschitz function y belonging to the convex set

$$\mathcal{Y} = \{y \in C^{0,1}([0,S]) : y(0) = 0\,, \|y'\|_\infty \le C\}\,, \tag{1}$$

for $S \ll 1$ and $C \gg 1$. We will focus on finding the path of propagation and not on its parametrization in time. For $y \in C^{0,1}$ it is not known whether the stress intensity factors exist; however, we known that the elastic field has a higher integrability property: there exists $2 < q < 4$ such that for every $y \in \mathcal{Y}$ and every $s \in [0,S]$ the equilibrium configuration, say u_s, belongs to $W^{1,q}(\Omega \setminus \Gamma_s)$. Thanks to this property we can define the following quantities

$$\widetilde{K}_i(\Gamma_s, \phi) = \int_{B_r \setminus \Gamma_s} (u_s - \mathring{u}_s) \cdot k_i(\theta - \phi)\, dx\,,$$

where $\mathring{u}_s$ denotes the displacement of the crack tip, θ is the argument in a system of polar coordinates centered at the crack tip and translating rigidly, $\phi \in (-\pi, \pi)$ and k_i is a suitable convolution kernel. When the SIF exist and when $\phi = \arctan(y')$ then $\widetilde{K}_i(\Gamma_s, \phi)$ provides an approximation, denoted by $\widetilde{K}_i(\Gamma_s)$, of the true SIF $K_i(\Gamma_s)$. Moreover for every $s \in [0,S]$ there exists a unique angle ϕ_s such that $\tilde{K}_i(\Gamma_s, \phi_s) = 0$. Then, the crack path is found by solving the first order functional

differential equation

$$\begin{cases} y'(s) = \tan\phi_s & \text{for a.e. } s \in (0,S) \\ y(0) = 0\,. \end{cases}$$

Indeed, if $y'(s) = \tan\phi_s$ then $\widetilde{K}_{\mathrm{II}}(\Gamma_s, \phi_s) = \widetilde{K}_{\mathrm{II}}(\Gamma_s) = 0$. In this way, the path will satisfy the Principle of Local Symmetry, in approximated form, for a.e. $s \in [0,S]$. Moreover, the solution is of class $C^{1,\alpha}([0,S])$ for $\alpha = 1 - 2/q$ and q the higher integrability exponent. Therefore, $\widetilde{K}_{\mathrm{II}}(\Gamma_s) = 0$ for every $s \in (0,S]$.

References

[1] J. Ball Some open problems in elasticity. *Geometry, mechanics, and dynamics* 3–59, Springer, New York, 2002.

[2] A. Chambolle, G. Francfort and J.-J. Marigo When and how do cracks propagate? *J. Mech. Phys. Solids* 57(9):1614–1622, 2009.

[3] B. Cotterell On brittle fracture paths. *Internat. J. Fracture*, 1:96–103, 1965.

[4] R.V. Goldstein and R.L. Salganik. Brittle fracture of solids with arbitrary cracks. *Internat. J. Fracture*, 10:507–523, 1974.

[5] P. Grisvard. Singularités en elasticité. *Arch. Rational Mech. Anal.*, 107(2):157–180, 1989.

[6] G. Lazzaroni, R. Toader Energy release rate and stress intensity factor in antiplane elasticity *J. Math. Pures Appl.*, 95:565–584, 2011.

[7] M. Negri. A comparative analysis on variational models for quasi-static brittle crack propagation. *Adv. Calc. Var.*, 3:149–212, 2010.

[8] M.L. Williams. On the stress distribution at the base of a stationary crack. *J. Appl. Mech.*, 24:109–114, 1957.

Notions of differential calculus on metric measure spaces

Nicola Gigli

(joint work with Luigi Ambrosio and Giuseppe Savaré)

Aim of the talk is to present recent advances on abstract analysis over metric measure spaces, with a particular focus on those having Ricci curvature bounded from below in the sense of Lott-Sturm-Villani.

In particular, it will be shown how enhancing the curvature dimension condition with the requirement that the heat flow is linear, leads to a new class of spaces which is still stable w.r.t. measured Gromov-Hausdorff convergence, compatible with the Riemannian case, which rules out Finsler geometries.

Among other properties of these notion there are:

- exponential contractivity of the Wasserstein distance W_2 along two heat flows,
- full compatibility with the theory of Dirichlet forms,
- existence of a Brownian motion with continuous sample paths,
- validity of the Bakry-Emery curvature condition,
- in case the measure is doubling and supports a local Poincaré inequality: Lipschitz continuity of the heat kernel.

References

[1] Luigi Ambrosio, Nicola Gigli, Giuseppe Savaré, *Calculus and heat flow on metric measure spaces, with application to spaces with Ricci curvature bounded from below*, submitted.

[2] Luigi Ambrosio, Nicola Gigli, Giuseppe Savaré, *Metric measure spaces with Riemannian Ricci curvature bounded from below*, submitted.

The h-principle for the Euler equations

László Székelyhidi Jr.

In this talk I report on recent joint work with Camillo De Lellis. We consider the incompressible Euler equations on the torus $\mathbb{T}^3$, given by

$$\begin{aligned} \partial_t v + \operatorname{div} v \otimes v + \nabla p &= 0, \\ \operatorname{div} v &= 0. \end{aligned} \tag{1}$$

It is well known that classical solutions (say, $v \in C^1$) conserve the energy $E(t) = \frac{1}{2}\int |v|^2\,dx$. On the other hand in recent years there has been a lot of focus on understanding weak solutions, which do not necessarily conserve the energy. The motivation for this comes from the Kolmogorov-Onsager theory of homogeneous isotropic turbulence, laid down some 70 years ago.

In the formulation of Onsager the conjecture is that weak solutions of the Euler equations preserve the energy if the solution is Hölder-continuous with exponent $\alpha > 1/3$, whereas if $\alpha < 1/3$, weak solutions need not preserve the energy. Remarkably, the critical exponent $1/3$ matches exactly with exponents obtained by Kolmogorov for the decay of the energy spectrum in the inertial regime, even though Kolmogorov's calculations were for ensemble averages of solutions to the Navier-Stokes equations, based on the assumption that there is an energy cascade. In turn, the energy cascade links to an idea of Richardson concerning the structure of turbulent flows, which, roughly speaking, amounts to a cascade of vortex structures appearing in a self-similar fashion. Thus, the conjecture of Onsager, on the face of it a conjecture about optimal regularity, is linked to a (less precise) conjecture about a possible self-similar structure in weak solutions of the Euler equations. It is this problem that is addressed in the talk.

The first example of a weak solution of (1), which does not conserve the energy, is due to V. Scheffer in 1993, an example by A. Shnirelman was produced by a different method in 1997. It turns out, that the existence of such weak solutions can be understood when formulating (1) in the framework introduced by L. Tartar and R. DiPerna in the 1970s. Observing that the equations (1) are in form of a conservation law, we can formulate them as

$$\begin{aligned} \sum_{i=1}^{d} A_i \partial_i z &= 0, \\ z(y) &\in K \subset \mathbb{R}^N. \end{aligned} \tag{2}$$

By studying special one-dimensional oscillatory solutions (plane-waves) of the form

$$z(y) = \hat{z}\, e^{iy\cdot\xi}. \tag{3}$$

one obtains a *relaxation* of (2), taking the form

$$\begin{aligned} \sum_{i=1}^{d} A_i \partial_i z = 0, \\ z(y) \in K^{\Lambda}, \end{aligned} \tag{4}$$

where, in general, $K \subset K^{\Lambda} \subset K^{co}$. In the case of the Euler equations it turns out (see [1]) that $K^{\Lambda} = K^{co}$, in some sense meaning that the are many oscillations compatible with the Euler equations.

Conversely, it is known that in certain cases where K^{Λ} is sufficiently large, most solutions of (4) are in fact solutions of (2). Indeed, one can start from a solution of (4) and iteratively reintroduce localized versions of the plane-waves (3) in order to arrive at a solution of (2). The origin of this idea can be traced back to the landmark work of Nash on C^1-isometric embeddings and the generalization thereof by Gromov into what became known as convex integration. This method has been subsequently adapted to the situation (2), in particular for systems coming from nonlinear elasticity, by S. Müller and V. Šverák, B. Kirchheim, B. Dacorogna-P. Marcellini, and many others. The corresponding statement arising from this theory for (1) is:

Theorem 1 ([2])**.** *Let $\overline{e} \in C(\mathbb{T}^n \times (0,T))$. Then there exist infinitely many weak solutions $v \in L^{\infty}_{loc}(\mathbb{T}^n \times (0,T))$ of the Euler equations such that*

$$\tfrac{1}{2}|v|^2 = \overline{e} \text{ for a.e. } (x,t).$$

In the formulation of Theorem 1 we deliberately avoided mentioning the initial data. The reason for this stems from the fact, that, as has been argued in [1, 2], the Euler equations abide by Gromov's h-principle. A typical feature of the h-principle is that one can distinguish between a local and a global aspect. The local aspect would involve the passage from subsolutions to solutions, as in Theorem 1 (in other words the passage from (4) to (2)), whereas the global aspect involves matching initial/boundary conditions. Indeed, it is not difficult to extend the proof of Theorem 1 to produce counterexamples to well-posedness of the initial-value problem, as has been done in [2].

The basic idea of passing from (smooth) solutions of (4) to (rough) solutions of (2) by reintroducing oscillations has been first used by Nash in producing C^1 isometric immersions. In local coordinates the problem can be formulated as

$$\nabla u^T \nabla u = g,$$

which can again be cast in the form (2). Here, once again $K^{\Lambda} = K^{co}$, corresponding to *short* embeddings. The oscillations in this case take the very explicit form of spirals. Assuming that u is a smooth strictly short embedding, let η, ζ be normal

vector fields to the image of u, i.e. unit vector fields satisfying $\nabla u^T \eta = \nabla u^T \zeta = 0$. The perturbed embedding takes locally the form

$$\tilde{u}(x) = u(x) + \frac{a(x)}{\lambda}\left(\sin(\lambda\xi \cdot x)\eta(x) + \cos(\lambda\xi \cdot x)\zeta(x)\right)$$

It is easy to see that the perturbed metric has now the form

$$\nabla \tilde{u}^T \nabla u = \nabla u^T \nabla u + a^2 \xi \otimes \xi + O(\tfrac{1}{\lambda}). \tag{5}$$

The explicit form of the perturbation makes it possible to obtain a continuous gradient in the limit. In the generality of (2) this seems to be out of reach. Nevertheless, for the Euler equations (1) this turns out to be possible:

Theorem 2 ([3]). *Assume $\bar{e} : [0,1] \to \mathbb{R}$ is a positive smooth function. Then there is a continuous weak solution $v : \mathbb{T}^3 \times [0,1] \to \mathbf{R}^3$ of the Euler equations* (1) *such that*

$$\bar{e}(t) = \int |v|^2(x,t)\, dx \qquad \forall t \in [0,1]\,.$$

Though the proof shares several similarities with Nash's scheme, there are many points where the method departs dramatically from Nash's, due to some issues which are typical of the Euler equations and are not present for the isometric embeddings.

1) Perhaps the most important new aspect of our scheme is a "transport term" which arises, roughly speaking, as the linearization of (1): this term is typical of an evolution equation, whereas, instead, the equations for isometric embeddings are "static". At a first glance this transport term makes it impossible to use a scheme like the one of Nash. To overcome this obstruction we need to introduce a phase-function that acts as a kind of discrete Galilean transformation of the (stationary) Beltrami flows, and to introduce an "intermediate" scale along each iteration step on which this transformation acts.

2) The convex integration scheme of Nash and Gromov heavily relies on one-dimensional oscillations - the simple reason being that these can be "integrated", hence the name convex integration. As already mentioned, the main building blocks of our iteration scheme are Beltrami flows, which are truly three-dimensional oscillations. The issue of going beyond one-dimensional oscillations has been raised by Gromov as well as Kirchheim-Müller-Šverák, but as far as we know, there have been no such examples in the literature so far. In fact, it seems that with one-dimensional oscillations alone one cannot overcome the obstruction in 1).

3) A third, more technical, new aspect is the absence of a simple potential to generate solutions of the Euler system: in a sense we cannot simply "integrate" Beltrami flows. In order to overcome this issue we introduce a "corrector term" to the main perturbation. In order to estimate the corrector and show that its contribution is $o(1)$ (analogously to the $O(\frac{1}{\lambda})$ term in (5)), we use a combination of standard Schauder theory and oscillatory integrals estimates. This gives to our proof a "hard" PDE flavor compared to the construction of Nash, which is more on the side of "soft analysis".

References

[1] De Lellis, C., and Székelyhidi, Jr., L. The Euler equations as a differential inclusion. *Ann. of Math. (2) 170*, 3 (2009), 1417–1436.

[2] De Lellis, C., and Székelyhidi, Jr., L. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal. 195*, 1 (2010), 225–260.

[3] De Lellis, C., and Székelyhidi, Jr., L. Dissipative continuous Euler flows. *Preprint* (2012).

Optimal-Transportation Meshfree Approximation Schemes

Michael Ortiz

(joint work with Bo Li, Bernd Schmidt)

Approximation schemes based on Optimal Transportation theory (e. g., [2]), when combined with with meshfree interpolation (cf., e. g., [1]) and *material point* sampling [3] supply a powerful alternative to the strictly Lagrangian or Eulerian paradigms that combines some of the best attributes of both. A simple case in point is furnished by Euler flows over time dependent domains, e. g., resulting from fluid-structure interaction. For these systems, inertia competes with free energy in determining the flow of mass. Conveniently, the free energy of a fluid can be expressed directly in terms of its mass density and the resulting action can also be expressed directly in terms of the mass density. A class of semi-discrete actions that is well-suited to computation is [3]

(1)
$$A_d(\rho_1,\dots,\rho_{N-1}) = \sum_{k=0}^{N-1} \left\{ \frac{1}{2}\frac{\mathcal{T}_2(\rho_k,\rho_{k+1})}{(t_{k+1}-t_k)^2} - \frac{1}{2}[U(\rho_k)+U(\rho_{k+1})] \right\} (t_{k+1}-t_k)\,,$$

which is expressed directly in terms of densities ρ_0, ρ_1, $\dots$, ρ_k, $\dots$ ρ_N at the discrete times t_0, t_1, $\dots$, t_k, $\dots$ t_N. In this expression, the functional $U(\rho)$ gives the total internal energy of the fluid and ρ_0 and ρ_N are presumed given. We note that, in (1), the inertial part of the action is given by the Wasserstein distance $\mathcal{T}_2(\rho_k,\rho_{k+1})$ between consecutive mass densities. Further extensions to solid flows, which requires consideration of more general actions, and to viscous and inelastic behavior are presented in [3]. The discrete equations of motion now follow by rendering the semi-discrete (1) action stationary. Taking appropriate variations of (1) gives [3]

(2)
$$\frac{2\rho_k}{t_{k+1}-t_{k-1}}\left(\frac{\varphi_{k\to k+1}-\mathrm{id}}{t_{k+1}-t_k}+\frac{\varphi_{k\to k-1}-\mathrm{id}}{t_k-t_{k-1}}\right) = \nabla p_k + \rho_k b_k\,,$$

where p_k and b_k are the pressure and body-force fields at time t_k, and $\varphi_{k\to k+1}$ is the transportation map, or incremental deformation mapping, between the configurations at time t_k and t_{k+1}. Evidently, the semi-discrete equations of motion (2) supply a central-difference-like time discretization of the equations of motion of the fluid.

In order to obtain a fully discrete action for computations, the semi-discrete action (1) needs to be discretized in space. The scope of this discretization is

three-fold and it concerns: i) the discretization of the volume measure $\mathcal{L}$; ii) the discretization of the mass densities ρ_k; and iii) the discretization of the incremental transportation maps $\varphi_{k\to k+1}$. We achieve the first of these spatial discretizations simply by approximating the usual Lebesgue measure $\mathcal{L}$ by discrete measures of the form

$$\mathcal{L}_{h,k} = \sum_{p=1}^{M} \upsilon_{p,k}\, \delta_{x_{p,k}}\,, \tag{3}$$

concentrated at *material points* $x_{p,k}$, each of which is assigned a discrete volume $\upsilon_{p,k}$. We achieve the second spatial discretization simply by identifying the discrete mass distributions as measures that are absolutely continuous with respect to the discrete volume measure $\mathcal{L}_{h,k}$, with Radon-Nykodim density $\rho_{h,k}$, i. e.,

$$\rho_{h,k}(x) = \sum_{p=1}^{M} \rho_{p,k}\upsilon_{p,k}\delta\big(x - x_{p,k}\big)\,. \tag{4}$$

The quantity $m_p = \rho_{p,k}\upsilon_{p,k}$ may be regarded as the mass *carried* by material point p. A weak reformulation of the continuity equation then shows that discrete mass is conserved if and only if m_p is constant and independent of time. Finally, the incremental transportation maps $\varphi_{k\to k+1}$ may be interpolated by mesh-free conforming interpolation of the form

$$\varphi_{h,k\to k+1}(x) = \sum_{a=1}^{N} x_{a,k+1} N_{a,k}(x), \tag{5}$$

where $\{x_{a,k+1},\ a = 1, \ldots, N\}$ are coordinates of *nodes* on the configuration at time t_{k+1} and $N_{a,k}(x)$ are conforming shape functions defined over the configuration at time t_k. The use of *max-ent meshfree interpolation* [1] is particularly appealing, as the incremental shape functions $N_{a,k}(x)$ are defined essentially explicitly in terms of the nodal coordinates $\{x_{a,k},\ a = 1, \ldots, N\}$ at time t_k. By continuously updating these shape functions, unconfined flows can be computed free of mesh-entanglement obstructions. This combination of optimal transportation and meshfree interpolation has been termed *Optimal-Transportation Meshfree* (OTM) approximation in [3].

Applications of OTM approximation to fluid-structure interaction problems and extensions to plastic solid flows may be found in [3]. A particularly appealing feature of OTM approximation is that it results in *geometrically exact* updates of mass and volume, without the need for solving a Poisson equation for the pressure or approximating advection equations. In addition, because the flow of computations is of the *updated Lagrangian* form, the coupling of fluids and solids or structures or the implementation of moving boundaries and free surfaces in fluid flows is trivial. These features, in conjunction with the mesh-free character of the interpolation, confers OTM approximation excellent properties of robustness and computational efficiency.

References

[1] M. Arroyo and M. Ortiz, *Local maximum-entropy approximation schemes: a seamless bridge between finite elements and meshfree methods*, International Journal for Numerical Methods in Engineering **65** (2006) 2167-2202.

[2] C. Villani, *Topics in optimal transportation theory*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, Rhode Island, USA, 2003.

[3] B. Li, F. Habbal and M. Ortiz, *Optimal transportation meshfree approximation schemes for fluid and plastic flows*, International Journal for Numerical Methods in Engineering **83** (2010) 1541-1579.

Stability and instability of asymptotic profiles for fast diffusion

Goro Akagi

(joint work with Ryuji Kajikiya)

Introduction. Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We are concerned with the Cauchy-Dirichlet problem for fast diffusion equations of the form

$$\partial_t \left(|u|^{m-2}u\right) = \Delta u \quad \text{in } \Omega \times (0,\infty), \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega, \tag{3}$$

where $\partial_t = \partial/\partial t$, $2 < m < 2^* := 2N/(N-2)_+$ and $u_0 \in H_0^1(\Omega)$. Every solution $u = u(x,t)$ of (1)–(3) for $u_0 \neq 0$ vanishes at a finite time $t_* > 0$ with the rate of $(t_* - t)^{1/(m-2)}$. The finite time $t_* = t_*(u_0)$ is called *extinction time* (of the unique solution u) for a data u_0. Then one can define the *asymptotic profile* $\phi = \phi(x)$ of each solution $u = u(x,t)$:

$$\phi(x) := \lim_{t \nearrow t_*} (t_* - t)^{-1/(m-2)} u(x,t) \quad \text{in } H_0^1(\Omega).$$

In order to characterize ϕ, we apply the following transformation:

$$v(x,s) := (t_* - t)^{-1/(m-2)} u(x,t) \quad \text{and} \quad s := \log(t_*/(t_* - t)) \geq 0. \tag{4}$$

Then the asymptotic profile $\phi = \phi(x)$ of $u = u(x,t)$ is reformulated as $\phi(x) := \lim_{s \nearrow \infty} v(x,s)$ in $H_0^1(\Omega)$. Moreover, the Cauchy-Dirichlet problem (1)–(3) for $u = u(x,t)$ is rewritten by

$$\partial_s \left(|v|^{m-2}v\right) = \Delta v + \lambda_m |v|^{m-2}v \quad \text{in } \Omega \times (0,\infty), \tag{5}$$

$$v = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{6}$$

$$v(\cdot,0) = v_0 \quad \text{in } \Omega, \tag{7}$$

where $v_0 = t_*(u_0)^{-1/(m-2)} u_0$ and $\lambda_m = (m-1)/(m-2) > 0$.

In a celebrated paper by Berryman and Holland [2] and several papers (see [3, 4]), the existence of asymptotic profiles for each solution for (1)–(3) was proved,

and moreover, every asymptotic profile turns out to be a nontrivial solution of the Dirichlet problem for the Emden-Fowler equation,

$$-\Delta\phi = \lambda_m|\phi|^{m-2}\phi \ \text{ in } \ \Omega, \quad \phi = 0 \ \text{ on } \ \partial\Omega. \tag{8}$$

Then one can also find that the set of all asymptotic profiles of solutions for (1)–(3) coincides with the set of all nontrivial solutions of (8). In this talk, we address ourselves to the stability/instability of asymptotic profiles.

Main results. We first introduce the notions of stability and instability of asymptotic profiles.

Definition 1 ([1])**.** Let $\phi \in H_0^1(\Omega)$ be an asymptotic profile of vanishing solutions.

(i) ϕ is said to be *stable*, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that any solution v of (5)–(7) satisfies

$$v(0) \in \mathcal{X} \cap B(\phi;\delta) \quad \Rightarrow \quad \sup_{s\in[0,\infty)} \|v(s) - \phi\|_{1,2} < \varepsilon$$

with a surface $\mathcal{X} := \{t_*(u_0)^{-1/(m-2)}u_0 \colon u_0 \in H_0^1(\Omega) \setminus \{0\}\}$.

(ii) ϕ is said to be *unstable*, if ϕ is not stable.

(iii) ϕ is said to be *asymptotically stable*, if ϕ is stable, and moreover, there exists $\delta_0 > 0$ such that any solution v of (5)–(7) satisfies

$$v(0) \in \mathcal{X} \cap B(\phi;\delta_0) \quad \Rightarrow \quad \lim_{s\nearrow\infty} \|v(s) - \phi\|_{1,2} = 0.$$

Let d_1 be the *least energy* of a functional J associated with (8) and given by

$$J(w) = \frac{1}{2}\|w\|_{1,2}^2 - \frac{\lambda_m}{m}\|w\|_m^m \quad \text{for } w \in H_0^1(\Omega)$$

over nontrivial solutions, i.e.,

$$d_1 := \inf_{v\in\mathcal{S}} J(v), \quad \mathcal{S} := \{ \text{ nontrivial solutions of (8)}\}.$$

A *least energy solution* ϕ_1 of (8) means $\phi_1 \in \mathcal{S}$ satisfying $J(\phi_1) = d_1$. We remark that every least energy solution of (8) is sign-definite by maximum principle. Then we present criteria for the stability and instability of asymptotic profiles.

Theorem 2 ([1])**.** *Let ϕ be a least energy solution of* (8)*. Then*

(i) *ϕ is a stable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the other least energy solutions.*

(ii) *ϕ is an asymptotically stable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the other sign-definite solutions.*

Theorem 3 ([1])**.** *Let ϕ be a sign-changing solution of* (8)*. Then*

(i) *ϕ is not an asymptotically stable profile.*

(ii) *ϕ is an unstable profile, if ϕ is isolated in $H_0^1(\Omega)$ from $\{\psi \in \mathcal{S} \colon J(\psi) < J(\phi)\}$.*

References

[1] G. Akagi and R. Kajikiya, *Stability analysis of asymptotic profiles for sign-changing solutions to fast diffusion equations*, submitted.
[2] J.G. Berryman and C.J. Holland, *Stability of the separable solution for fast diffusion*, Arch. Rational Mech. Anal. **74** (1980), 379–388.
[3] Y.C. Kwong, *Asymptotic behavior of a plasma type equation with finite extinction*, Arch. Rational Mech. Anal. **104** (1988), 277–294.
[4] G. Savaré and V. Vespri, *The asymptotic profile of solutions of a class of doubly nonlinear equations*, Nonlinear Anal. **22** (1994), 1553–1565.

Global solvability of the Landau-Lifshitz-Gilbert equation

Christof Melcher

For director fields $\boldsymbol{m} = (0,\infty) \times \mathbb{R}^n \to \mathbb{S}^2$ with values in the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ we consider the Landau-Lifshitz-Gilbert equation (LLG)

$$\frac{\partial \boldsymbol{m}}{\partial t} = -\boldsymbol{m} \times \Delta \boldsymbol{m} - \lambda\, \boldsymbol{m} \times \boldsymbol{m} \times \Delta \boldsymbol{m} \tag{1}$$

where λ is a positive damping parameter and $\times$ is the vector product on $\mathbb{R}^3$. This quasilinear evolution equation is a hybrid between the heat and Schrödinger flow for harmonic maps. As such it is neither a gradient flow nor a hamiltonian system, but with the Dirichlet energy $E(\boldsymbol{m}) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \boldsymbol{m}|^2\, dx$ still serving as a Liapunov functional. According to the natural parabolic scaling symmetry, the critical dimension is $n = 2$, where the topological lower bound $E(\boldsymbol{m}_0) < 4\pi$ for an initial map $\boldsymbol{m}_0$ implies global regularity, see [8, 3, 4]. In dimension $n \geq 3$, however, LLG becomes super-critical and arguments exclusively based on the energy are insufficient for ruling out concentration effects and the formation of finite time singularities. The regularity theory for the heat flow equation in higher dimensions relies heavily on specific features stemming from its gradient flow structure, in particular Bochner identities and Struwe's monotonicity formulas, see [9, 11]. These tools are not available for LLG, and auxiliary methods based on elliptic problems on suitable time slices are restricted to dimensions $n = 3, 4$, see [6, 7, 10]. In this talk we establish a relationship between LLG and a covariant complex Ginzburg-Landau equation with a nonlinearity that is essentially cubic. This observation is motivated by recent developments in the context of Schrödinger maps, see [1, 2]. The main point is that we have transformed the problem into a semilinear one. Inspired by Kato's weighted-in-time approach to the well-posedness of the Navier-Stokes equation in $L^n(\mathbb{R}^n)$, see [5], we were able to prove the following global existence, uniqueness and regularity result, valid in arbitrary dimensions:

Theorem. *Suppose $\lambda > 0$ and $n \geq 3$. Then there exist constants $\rho > 0$ and $c > 0$ with the following property: Given $\boldsymbol{m}_\infty \in \mathbb{S}^2$ and initial data $\boldsymbol{m}_0 : \mathbb{R}^n \to \mathbb{S}^2$ such that $\boldsymbol{m}_0 - \boldsymbol{m}_\infty \in H^1 \cap W^{1,n}(\mathbb{R}^n; \mathbb{R}^3)$ and such that*

$$\|\nabla \boldsymbol{m}_0\|_{L^n} < \rho,$$

then there exists a global smooth solution $\boldsymbol{m} : (0,\infty)\times\mathbb{R}^n \to \mathbb{S}^2$ *for the Landau-Lifshitz-Gilbert equation* (1) *with the properties that*

$$\sup_{t>0} \|\boldsymbol{m}(t) - \boldsymbol{m}_\infty\|_{H^1} \le \|\boldsymbol{m}_0 - \boldsymbol{m}_\infty\|_{H^1}$$

and

$$\sup_{t>0} \sqrt{t}\ \|\nabla \boldsymbol{m}(t)\|_{L^\infty} + \sup_{t>0} \|\nabla \boldsymbol{m}(t)\|_{L^n} \le c\, \|\nabla \boldsymbol{m}_0\|_{L^n}$$

and such that $\lim_{t\searrow 0} (\boldsymbol{m}(t) - \boldsymbol{m}_0) = 0$ *strongly in* $H^1 \cap W^{1,n}(\mathbb{R}^n;\mathbb{R}^3)$. *The solution is unique in its class and satifies* $\lim_{t\to\infty} \boldsymbol{m}(t) = \boldsymbol{m}_\infty$ *in* $C^1(\mathbb{R}^n;\mathbb{R}^3)$.

References

[1] Bejenaru, I. and Ionescu, A. D. and Kenig, C. E., *Global existence and uniqueness of Schrödinger maps in dimensions* $d \ge 4$, Advances in Mathematics **1** (2007), 263–291.

[2] Bejenaru, I. and Ionescu, A. D. and Kenig, C. E. and D. Tataru, *Global Schrödinger maps*, Annals Math. (to appear)

[3] Guo, Bo Ling, Hong, Min Chun, *The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps* Calc. Var. Partial Differential Equations **3** (1993), 311–334.

[4] Harpes, Paul, *Uniqueness and bubbling of the 2-dimensional Landau-Lifshitz flow* Calc. Var. Partial Differential Equations **2** (2004), 213–229.

[5] , Kato, Tosio *Strong* L^p*-solutions of the Navier-Stokes equation in* $\mathbf{R}^m$*, with applications to weak solutions*, Math. Z. **187** (1984), 471–480.

[6] Melcher, Christof, *Existence of partially regular solutions for Landau-Lifshitz equations in* $\mathbb{R}^3$ Comm. Partial Differential Equations **30** (2005), 567–587.

[7] Moser, Roger, *Partial regularity for harmonic maps and related problems* (2005).

[8] Struwe, Michael, *On the evolution of harmonic mappings of Riemannian surfaces* Comment. Math. Helv., **60** (1985).

[9] Struwe, Michael, *On the evolution of harmonic maps in higher dimensions*, J. Differential Geom. **28** (1988).

[10] Wang, Changyou, *On Landau-Lifshitz equation in dimensions at most four*, Indiana Univ. Math. J. **55** (2006), 1615–1644.

[11] Wang, Changyou, *Heat flow of harmonic maps whose gradients belong to* $L^n_x L^\infty_t$, Arch. Ration. Mech. Anal., **188** (2008), 351–369.

Remarks on the limit of the Cahn-Hilliard equation in 1D

Giovanni Bellettini

(joint work with L. Bertini, M. Mariani and M. Novaga)

Given $\epsilon \in (0,1]$, let us denote by $u_\epsilon = u_\epsilon(\cdot,\cdot,\overline{u})$ the solution to the Cahn-Hilliard equation

$$\begin{cases} u_t = \big(W'(u) - \epsilon^2 u_{xx}\big)_{xx} & \text{in } (0,+\infty)\times\mathbb{T}, \\ u = \overline{u} & \text{on } \{0\}\times\mathbb{T}, \end{cases} \tag{1}$$

where $\mathbb{T}$ is the one-dimensional torus, $W(z) = \frac{1}{4}(1-z^2)^2$ for $z\in\mathbb{R}$, and the initial datum $\overline{u}$ is of class $\mathcal{C}^1(\mathbb{T})$. This equation is the gradient flow, in the $H^{-1}(\mathbb{T})$-metric,

of the functional

$$F_\epsilon(v) := \int_{\mathbb{T}} \left(\epsilon^2 \frac{(v_x)^2}{2} + W(v) \right) \, dx. \tag{2}$$

We are interested in the existence of the limit (if any) as $\epsilon \downarrow 0$ of $u_\epsilon(\cdot, \cdot, \overline{u})$, which is a question posed in [3] for another equation, but similar in spirit to the one considered here. We are not able to answer this question, and we will analyze, instead, the limit of $u_\epsilon(\cdot, \cdot, (\overline{u}_\epsilon))$ for suitable sequences $(\overline{u}_\epsilon)$ of initial data converging to $\overline{u}$. The main difficulty is the forward-backward parabolic character of the equation in (1) when considered *formally* for $\epsilon = 0$, namely of the gradient flow of the nonconvex functional

$$F(v) := \int_{\mathbb{T}} W(v) \, dx.$$

Following [4], denote by

$$\Sigma_L(\overline{u}) := \{x \in \mathbb{T} : W''(\overline{u}(x)) < 0\}$$

the *local unstable* region of $\overline{u}$, by W^{**} the convex envelope of W, and let also

$$\Sigma_G(\overline{u}) := \{x \in \mathbb{T} : W(\overline{u}(x)) > W^{**}(\overline{u}(x))\}$$

be the *global unstable* region of $\overline{u}$. The interesting initial data are those for which $\Sigma_L(\overline{u}) \neq \emptyset$. Indeed, when $\mathrm{dist}(\overline{u}(\mathbb{T}), \Sigma_L(\overline{u})) > 0$, the gradient flow equation of F is forward parabolic, and therefore gives raise to a global smooth solution, *different in general*, when $\Sigma_G(\overline{u}) \neq \emptyset$, from the solution

$$\begin{cases} \partial_t u = \left(W^{**\prime}(u)\right)_{xx} & \text{in } (0, +\infty) \times \mathbb{T}, \\ u = \overline{u} & \text{on } \{0\} \times \mathbb{T} \end{cases} \tag{3}$$

obtained as the gradient flow of the convexified functional

$$F^{**}(v) := \int_{\mathbb{T}} W^{**}(v) \, dx. \tag{4}$$

For an initial datum $\overline{u}$ with $\Sigma_L(\overline{u}) \neq \emptyset$, numerical experiments [2] show the quick formation of a microstructure in the graph of $u_\epsilon(\cdot, \cdot, \overline{u})$, actually *only* in $\Sigma_L(\overline{u})$, and *not* on the whole of $\Sigma_G(\overline{u})$ [1]. Still following the heuristic arguments in [2], let us now artificially superimpose to $\overline{u}$ a microstructure in a region Σ which satisfies

$$\Sigma \subseteq \Sigma_G(\overline{u}) \setminus \Sigma_L(\overline{u}), \tag{5}$$

corresponding to a sequence $(\widetilde{u}_\epsilon)$ of initial data approximating $\overline{u}$. Denoting by $u_\epsilon(\cdot, \cdot, (\widetilde{u}_\epsilon))$ the solution of (1), the numerical observations lead to believe that the limit solution depends on Σ and, in general, differs from the solution obtained as a limit (if any) of $u_\epsilon(\cdot, \cdot, \overline{u})$. In conclusion, considering two sequences $(\widetilde{u}_\epsilon)$, $(\widehat{u}_\epsilon)$ of initial data both approximating $\overline{u}$, and corresponding to different choices of Σ

[1] This remarkable observation appeared in the paper [4], where a a different ill-posed problem, and a different regularization, were considered.

satisfying (5), it seems that one cannot exclude that (assuming the existence of the limits)

$$\lim_{\epsilon\downarrow 0} u_\epsilon(\cdot,\cdot,(\widetilde{u}_\epsilon)) \;\neq\; \lim_{\epsilon\downarrow 0} u_\epsilon(\cdot,\cdot,(\widehat{u}_\epsilon)).$$

Our main result consists in the assertion that such limits can be characterized, when $\Sigma_L(\overline{u}) \neq \emptyset$, under the assumption (6) that $\overline{u}$ is energetically well-prepared (heuristically, this corresponds to take $\Sigma = \Sigma_G(\overline{u}) \setminus \Sigma_L(\overline{u})$ in (5)). The statement is the following.

Theorem. Let $\overline{u}$ be such that $F(\overline{u}) < +\infty$. Take a sequence $(\overline{u}_\epsilon)$ of initial data satisfying $F_\epsilon(\overline{u}_\epsilon) < +\infty$, $\int_{\mathbb{T}} \overline{u}_\epsilon \, dx = \int_{\mathbb{T}} \overline{u} \, dx$, converging to $\overline{u}$ and such that

$$\lim_{\epsilon\downarrow 0} F_\epsilon(\overline{u}_\epsilon) = F^{**}(\overline{u}). \tag{6}$$

Then the corresponding solutions $u_\epsilon(\cdot,\cdot,(\overline{u}_\epsilon))$ of (1) converge to the solution u of (3).

The proof of this result is entirely variational, and it is based on some ideas formalized in [6] (see also [5] for another application of these techniques) which reduce, in some sense, the asymptotic analysis of the sequence (u_ϵ) to the computation of the Γ-limsup of the sequence (F_ϵ), and to the Γ-liminf of the sequence of slopes $(|\nabla F_\epsilon|)$ of F_ϵ (as defined, for instance, in[1]).

References

[1] L. Ambrosio, N. Gigli, G. Savarè, Gradient Flows in Metric Spaces and in the Space of Probability Measures. Second edition, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel 2008.

[2] G. Bellettini, G. Fusco, N. Guglielmi, A concept of solution for forward-backward equations of the form $u_t = \frac{1}{2}(\phi'(u_x))_x$ and numerical experiments for the singular perturbation $u_t = -\epsilon^2 u_{xxxx} + \phi'(u_x)_x$, Discrete Contin. Dyn. Syst. **16** (2006), 783–842.

[3] E. De Giorgi, Congetture su alcuni problemi di evoluzione, Duke Math. J. **81** (1995), 255-268.

[4] F. Fierro, R. Goglione, M. Paolini, Numerical simulations of mean curvature flow in presence of a nonconvex anisotropy, Math. Mod. Meth. Appl. Sci. **8** (1998), 574–601.

[5] N.Q. Le, On the convergence of the Ohta-Kawasaki equation to motion by nonlocal Mullins-Sekerka law, SIAM J. Math. Anal. **42** (2010), 1602-1638.

[6] E. Sandier, S. Serfaty, Γ-convergence of gradient flows with applications to Ginzburg-Landau, Comm. Pure Appl. Math. **57** (2004), 1627-1672.

A shape optimization problem for Moreau's sweeping process

Giovanni Colombo

The sweeping process is the evolution inclusion

$$\begin{cases} \dot{x}(t) & \in -N_{C(t)}(x(t)) \\ x(0) & = x_0 \in C(0), \end{cases} \tag{1}$$

where $t \in [0,T]$, $t \to C(t)$ is a moving set in a Hilbert space, and $N_{C(t)}(x)$ denotes (a suitable) normal cone. Under natural assumptions ($t \to C(t)$ Lipschitz, $C(t)$

closed and convex for all t) it is well known, see, e.g., [3, 4, 2], that (1) admits one and only one solution in $[0,T]$, which is Lipschitz continuous.

We formulate and study an optimal control problem for the sweeping process, where control functions enter the moving set, considering a cost functional J depending on the trajectory of (1). To the best of our knowledge, this is the first study in the literature devoted to such problem (see also the partially related paper [5]). We first establish an existence theorem of optimal solutions in the case where $C(t) = \{x : u(t,x) \leq 0\}$, where $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is Lipschitz with respect to t and convex with respect to x, under suitable *a priori* bounds on the unknown u and natural assumptions on J. Next we derive necessary optimality conditions in the particular case where u is affine with respect to x. Our approach to necessary conditions is based on the method of discrete approximations and advanced tools of variational analysis and generalized differentiation. The final results obtained are given in terms of the initial data of the controlled sweeping process and are illustrated by some examples. In particular, if the cost functional is the distance of the final point from a given closed set, optimal trajectories may be explicitly computed.

The talk is based on [1].

References

[1] Colombo, G.; Henrion, R.; Mordukhovich, B.; Nguyen, Hoang D.: Optimal control of the sweeping process, DCDIS - B 19 (2012), 117-159

[2] Monteiro Marques, M. D. P.: *Differential Inclusions in Nonsmooth Mechanical Problems: Shocks and Dry Friction*, Birkhäuser, Boston, 1993.

[3] Moreau, J.-J.: On unilateral constraints, friction and plasticity, in *New Variational Techniques in Mathematical Physics*, Proceedings from CIME (G. Capriz, G. Stampacchia, eds.), Cremonese, Rome (1974), 173-322.

[4] Moreau, J.-J.: Evolution problem associated with a moving convex set in a Hilbert space, *J. Diff. Eqs.* **26** (1977), 347-374.

[5] Rindler, F.: Optimal control for nonconvex rate-independent evolution processes, *SIAM J. Control Optim.* **47** (2008), 2773–2794

Reporter: Matthias Liero

Participants

Virginia Agostiniani
SISSA
International School for Advanced Studies
Via Beirut n. 2-4
I-34014 Trieste

Prof. Dr. Goro Akagi
Graduate School of Science and Technology
Kobe University
Rokko-dai 1-1, Nada-ku
Kobe 657-8501
JAPAN

Prof. Dr. Luigi Ambrosio
Scuola Normale Superiore
Piazza dei Cavalieri, 7
I-56100 Pisa

Prof. Giovanni Bellettini
Dipartimento di Matematica
Universita di Roma Tor Vergata
Via della Ricerca Scientif. 1
I-00133 Roma

Prof. Andrea Braides
Dipartimento di Matematica
Universita di Roma Tor Vergata
Via della Ricerca Scientif. 1
I-00133 Roma

Prof. Dr. Yann Brenier
Laboratoire J.-A. Dieudonne
UMR CNRS 6621
Universite de Nice Sophia-Antipolis
Parc Valrose
F-06108 Nice Cedex 2

Prof. Dr. Dorin Bucur
Laboratoire de Mathematiques
Universite de Savoie
F-73376 Le Bourget du Lac Cedex

Prof. Dr. Jose Antonio Carrillo
ICREA and Departament de Matematiques
Universitat Autonoma de Barcelona
E-08193 Bellaterra

Prof. Dr. Antonin Chambolle
Centre de Mathematiques Appliquees
Ecole Polytechnique
F-91128 Palaiseau Cedex

Prof. Dr. Giovanni Colombo
Dipartimento di Matematica
Universita di Padova
Via Trieste, 63
I-35121 Padova

Prof. Dr. Gianni Dal Maso
S.I.S.S.A.
Via Bonomea 265
I-34136 Trieste

Prof. Dr. Camillo De Lellis
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Guido de Philippis
Scuola Normale Superiore di Pisa
Piazza dei Cavalieri 7
I-56126 Pisa

Prof. Dr. Jean Dolbeault
CEREMADE
Universite Paris Dauphine
Place du Marechal de Lattre de
Tassigny
F-75775 Paris Cedex 16

Prof. Dr. Gilles Francfort
LAGA UMR 7539
Institut Galilee
Universite Paris 13
Avenue Jean-Baptiste Clement
F-93430 Villetaneuse

Prof. Dr. Adriana Garroni
Dipartimento di Matematica
"Guido Castelnuovo"
Universita di Roma "La Sapienza"
Piazzale Aldo Moro, 2
I-00185 Roma

Dr. Nicola Gigli
Laboratoire J.-A. Dieudonne
Universite de Nice
Sophia Antipolis
Parc Valrose
F-06108 Nice Cedex 2

Manuel Gnann
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig

Dr. Karoline Götze
Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstr. 39
10117 Berlin

Dr. Sebastian Heinz
Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstr. 39
10117 Berlin

Prof. Dr. Michael Herrmann
Fachbereich Mathematik - FB 9
Universität des Saarlandes
Gebäude E2 4
Postfach 151150
66041 Saarbrücken

Prof. Dr. Ansgar Jüngel
Institut für Analysis, Technische Math.
und Versicherungsmathematik
Technische Universität Wien
Wiedner Hauptstr. 8 - 10
A-1040 Wien

Dr. Dorothee Knees
Weierstrass-Institute for Applied
Analysis and Stochastics
Mohrenstr. 39
10117 Berlin

Dr. Matthias Kurzke
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn

Prof. Dr. Christopher J. Larsen
Department of Mathematics
Worcester Polytechnic Institute
100 Institute Road
Worcester , MA 01609
USA

Matthias Liero
Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstr. 39
10117 Berlin

Dr. Stefano Lisini
Dipartimento di Matematica
Universita di Pavia
Via Ferrata, 1
I-27100 Pavia

Prof. Dr. Stephan Luckhaus
Mathematisches Institut
Universität Leipzig
Johannisgasse 26
04103 Leipzig

Dr. Edoardo Mainini
Dipartimento di Matematica
Universita di Pavia
Via Ferrata, 1
I-27100 Pavia

Prof. Dr. Daniel Matthes
Zentrum Mathematik
Technische Universität München
Boltzmannstr. 3
85747 Garching bei München

Dario C.S. Mazzoleni
Dipartimento di Matematica
Universita di Pavia
Via Ferrata, 1
I-27100 Pavia

Prof. Dr. Christof Melcher
Lehrstuhl I für Mathematik
RWTH Aachen
Wüllnerstr. 5b
52062 Aachen

Dr. Georg Menz
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig

Prof. Dr. Alexander Mielke
Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstr. 39
10117 Berlin

Dr. Matteo Negri
Dipartimento di Matematica
Universita di Pavia
Via Ferrata, 1
I-27100 Pavia

Prof. Dr. Michael Ortiz
Division of Engineering and
Applied Sciences; MS 104-44
California Institute of Technology
Pasadena , CA 91125
USA

Prof. Dr. Felix Otto
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig

Prof. Dr. Mark A. Peletier
Mathematical Sciences
University of Bath
Claverton Down
GB-Bath BA2 7AY

Prof. Dr. Errico Presutti
Dipartimento di Matematica
Universita degli Studi di Roma II
Tor Vergata
Via della Ricerca Scientifica
I-00133 Roma

Filip Rindler
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 0WB

Dr. Tomas Roubicek
Mathematical Institute
Charles University
Sokolovska 83
186 75 Praha 8
CZECH REPUBLIC

Prof. Dr. Etienne Sandier
Departement de Mathematiques
Faculte des Sciences
Universite de Tours
Parc de Grandmont
F-37200 Tours

Prof. Dr. Giuseppe Savare
Dipartimento di Matematica
"F. Casorati"
Universita di Pavia
Via Ferrata, 1
I-27100 Pavia

Dr. Antonio Segatti
Dipartimento di Matematica
Universita di Pavia
Via Ferrata, 1
I-27100 Pavia

Christian Seis
Department of Mathematics
University of Toronto
40 St George Street
Toronto , Ont. M5S 2E4
CANADA

Prof. Dr. Dejan Slepcev
Department of Mathematical Sciences
Carnegie Mellon University
5000 Forbes Ave.
Pittsburgh , PA 15213-3890
USA

Dr. Emanuele Nunzio Spadaro
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig

Dr. Ulisse Stefanelli
IMATI - CNR
Via Ferrata, 1
I-27100 Pavia

Prof. Dr. Karl-Theodor Sturm
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn

Prof. Dr. Laszlo Szekelyhidi
Mathematisches Institut
Universität Leipzig
Johannisgasse 26
04103 Leipzig

Dr. Florian Theil
Mathematics Institute
University of Warwick
Zeeman Building
GB-Coventry CV4 7AL