Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Forced frequency locking in S^1–equivariant differential equations

Lutz Recke, Daniela Peterhof

submitted: 9th August 1996

Weierstraß–Institut
für Angewandte Analysis
und Stochastik
Mohrenstraße 39
D – 10117 Berlin
Germany

Preprint No. 257
Berlin 1996

1991 Mathematics Subject Classification. 58F35, 34A47, 34C25, 34C27, 34D35.
Key words and phrases. Forced symmetry breaking, forced frequency locking, bifurcation from solution orbits, rotating waves, modulated waves.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
e–mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
e–mail (Internet): preprint@wias-berlin.de

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Abstract

The aim of this paper is to present a simple analytic stategy for predicting, or engineering, two frequency locking phenomena for $\mathbf{S}^1$-equivariant ordinary differential equations. First we consider the forced frequency locking of a rotating wave solution of the unforced equation with a forcing of "rotating wave type", and we describe the creation of modulated wave solutions which is connected with this locking phenomenon. And second, we consider the forced frequency locking of a modulated wave solution with a forcing of "modulated wave type". Especially, we describe the sets of all control parameters and of all forcings such that frequency locking occures, the dynamic stability and the asymptotic behavior (for the forcing intensity tending to zero) of the locked solutions and the structural stability of all the phenomena.

This paper is essentially founded on results from our previous work [41] concerning abstract forced symmetry breaking. The equations considered in the present paper are finite dimensional prototypes of certain infinite dimensional models describing the behavior of continuous wave operated or self-pulsating multisection DFB lasers under continuous or pulsating light injection, respectively.

1 Introduction

In this paper we consider forced (or "induced") frequency locking for $\mathbf{S}^1$-equivariant ordinary differential equations of the type

$$\dot{\xi}(t) = f(\xi(t), \lambda) - \eta(t). \tag{1.1}$$

In (1.1), $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is a smooth parameter depending vector field, and we suppose $S(e^{i\gamma})f(\xi, \lambda) = f(S(e^{i\gamma})\xi, \lambda)$ for all $\gamma \in \mathbb{R}$, $\xi \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}^n$, where S is an $\mathbf{S}^1$-representation on the ξ-space. Thus, λ is an "internal, symmetry preserving" control parameter, and $\eta(t)$ is an "external" control parameter (varying in a certain function space) which breaks the symmetry and the autonomy of the equation.

The aim of this work is to present a simple analytic strategy for predicting, or engineering, forced frequency locking for (1.1) in two cases: First we describe frequency locking of a rotating wave solution of the unforced equation with a forcing of "rotating wave type", and second we consider frequency locking of a modulated wave solution of the unforced equation with a forcing of "modulated wave type". The strategy is an application of our results on abstract forced symmetry breaking [41], which are founded, in their turn, on a Liapunov-Schmidt reduction, certain scaling techniques (Hadamard's lemma) and the Implicit Function Theorem and which are further developments of results of E. Dancer, J. K. Hale, P. Táboas and A. Vanderbauwhede.

The paper is organized as follows.

In Section 2 we briefly sum up some results on abstract forced symmetry breaking from our previous work [41].

In Section 3 we introduce notation and assumptions which will be used in the subsequent Sections 4 and 5.

In Section 4 we suppose the unperturbed equation

$$\dot{\xi}(t) = f(\xi(t), \lambda_0) \tag{1.2}$$

to have an orbitally stable rotating wave solution $\xi_0(t) = S(e^{i\alpha_0 t})x_0$, and we describe the frequency locking of this solution to a forcing $\eta(t) = S(e^{i\alpha t})y$ with $\alpha \approx \alpha_0$ and $y \approx 0$. We show that for small forcings (i.e. for small $\|y\|$) near the rotating wave solution occures a modulated wave solution (which is stable in the sense of (4.6)), which has a modulation frequency near $|\alpha - \alpha_0|$, and the modulation oscillation $\max\{\, \|\xi(t)\| : t \in \mathbb{R}\,\} - \min\{\, \|\xi(t)\| : t \in \mathbb{R}\,\}$ of which tends to zero for $\|y\|$ tending to zero. If the forcing increases then the modulation oscillation increases, too, but the modulation frequency decreases. Moreover, at a certain value of the forcing intensity the modulation frequency vanishes, and the modulated wave solution generically changes "back" into a finite, even number of rotating wave solutions, which are close to fixed phase shifts of the "initial" rotating wave solution $\xi_0(t)$ and which have exactly the same frequency as the forcing (saddle node bifurcations of rotating waves). We describe which of them are stable and which are unstable. In this sense, frequency locking of the rotating wave solution of the unperturbed equation (1.2) with a forcing $\eta(t) = S(e^{i\alpha t})y$ of "rotating wave type" occures. If the intensity of the forcing is increased further then again saddle node bifurcations of the rotating wave solutions (into a modulated wave solution) may occure or not. This depends on whether or not the locking cone is "lop-sided" (see Section 4). We describe this bifurcation scenario rigorously and uniformly for all control parameters $\lambda \approx \lambda_0$, $\alpha \approx \alpha_0$ and $y = \epsilon z$ with $\epsilon \in \mathbb{R}$ near zero and $z \in \mathbb{R}^m$ near z_0, where z_0 is a "nondegenerate direction" in $\mathbb{R}^m$ (i.e. such that the corresponding reduced bifurcation equation has nondegenerate solutions).

In Section 5 we suppose equation (1.2) to have an $\mathbf{S}^1$-orbitally stable (cf. Definition 4.2) modulated wave solution $\xi_0(t) = S(e^{i\alpha_0 t})x_0(t)$ with $x_0(t) = x_0(t+\frac{2\pi}{\beta_0})$ for all t, and we describe the quasiperiodic frequency locking of this solution to a forcing $\eta(t) = S(e^{i\alpha t})y(t)$ with $y(t) = y(t+\frac{2\pi}{\beta})$ and $y(t) \approx 0$ for all t, $\alpha \approx \alpha_0$ and $\beta \approx \beta_0$.

The motivation for our investigations comes from problems in semiconductor laser modeling. At present, self-pulsations (i.e. periodic intensity change in the output power with frequencies of tenth of gigahertz, cf., e.g., [37, 44, 7, 54, 53, 6]) and frequency locking of self-pulsations to optically injected modulations (cf. [4, 20, 33, 45]) are topics of intensive experimental and theoretical research. The mathematical models are, as a rule, ordinary differential equations (rate equations for the carrier densities) which are nonlinearly coupled with boundary value problems for dissipative hyperbolic systems of first order partial differential equations ("coupled mode equations" for the complex amplitudes of the electric field). Moreover, the models are equivariant with respect to an $\mathbf{S}^1$-representation on the state space ($e^{i\gamma} \in \mathbf{S}^1$ works trivially on the carrier densities and by multiplication on the complex amplitudes).

By means of the results of [41], the forced frequency locking behavior of these models can be described to a great extent by analogy with the description of the forced frequency locking behavior of $\mathbf{S}^1$-equivariant ordinary differential equations (which is presented here). The frequencies α and α_0 (resp. β and β_0) are the so-called optical or carrier frequencies (resp. the power frequencies) of the external light signal and the self-pulsation, respectively, and the internal, symmetry preserving control parameter λ describes the internal laser parameter (laser currents, geometric and material parameters, facet reflectivities), for details see [38].

2 Forced Symmetry Breaking for Abstract $\mathbf{T}^n$-Equivariant Equations

In this section we briefly sum up some results on abstract forced symmetry breaking from our previous work [41] which will be used in the subsequent Sections 4 and 5. For related work see the results of J. K. HALE and P. TÁBOAS [28, 30, 31, 47], [29, Section 17], [14, Chapter 11.4], A. VANDERBAUWHEDE [48, 50], [49, Chapter 8], E. DANCER [16, 17, 18, 19] and D. CHILLINGWORTH, J. MARSDEN and Y. H. WAN [12, 52, 10, 11].

Let X and $\tilde{X}$ be Banach spaces and Λ and Y normed vector spaces such that X and Y are continuously embedded into $\tilde{X}$. Further, let $F : X \times \Lambda \to \tilde{X}$ be a C^k-map (with $k \geq 2$) and $x_0 \in X$ and $\lambda_0 \in \Lambda$ points such that

$$F(x_0, \lambda_0) = 0, \tag{2.1}$$

$$\partial_x F(x_0, \lambda_0) \text{ is a Fredholm operator from } X \text{ into } \tilde{X}, \tag{2.2}$$

$$\tilde{X} = \ker \partial_x F(x_0, \lambda_0) \oplus \operatorname{im} \partial_x F(x_0, \lambda_0). \tag{2.3}$$

Further, let Λ_1 and Λ_2 be closed subspaces of Λ such that $\Lambda = \Lambda_1 \oplus \Lambda_2$ and that

$$\dim \Lambda_2 = n, \quad \partial_\lambda F(x_0, \lambda_0)\Lambda_2 = \ker \partial_x F(x_0, \lambda_0). \tag{2.4}$$

By $\lambda_{0j} \in \Lambda_j$ $(j = 1, 2)$ we denote the components of λ_0 with respect to this decomposition of Λ, i.e. $\lambda_0 = \lambda_{01} + \lambda_{02}$.

Finally, let $\mathbf{T}^n = \mathbf{S}^1 \times ... \times \mathbf{S}^1$ (n times) be the n-dimensional torus group and $T : \mathbf{T}^n \to \mathcal{L}(\tilde{X})$ a representation of $\mathbf{T}^n$ on $\tilde{X}$ such that for all $\gamma \in \mathbf{T}^n$ we have

$$T(\gamma)X \subseteq X, \; T(\gamma)Y \subseteq Y, \tag{2.5}$$

$$F(T(\gamma)x, \lambda) = T(\gamma)F(x, \lambda) \text{ for all } x \in X \text{ and } \lambda \in \Lambda. \tag{2.6}$$

We suppose that

$$\gamma \in \mathbf{T}^n \longmapsto (T(\gamma)x, T(\gamma)\tilde{x}) \in X \times \tilde{X} \text{ is continuous for all } x \in X \text{ and } \tilde{x} \in \tilde{X}, \tag{2.7}$$

$$(\gamma, y) \in \mathbf{T}^n \times Y \longmapsto T(\gamma)y \in \tilde{X} \text{ is } C^k\text{-smooth}. \tag{2.8}$$

For $\tilde{x} \in \tilde{X}$ we denote by $\mathcal{O}(\tilde{x}) := \{T(\gamma)\tilde{x} : \gamma \in \mathbf{T}^n\}$ and by $\Gamma(\tilde{x}) := \{\gamma \in \mathbf{T}^n : T(\gamma)\tilde{x} = \tilde{x}\}$ the group orbit and the isotropy subgroup of $\tilde{x}$ with respect to T, respectively. We suppose

$$\dim \ker \partial_x F(x_0, \lambda_0) = n, \quad \dim \Gamma(x_0) = 0. \tag{2.9}$$

The following results concerning the abstract forced symmetry breaking problem

$$F(x, \lambda) = y, \quad x \approx \mathcal{O}(x_0), \ \lambda \approx \lambda_0, \ y \approx 0 \tag{2.10}$$

are proved in [41].

The first theorem describes the solution behavior of the "unperturbed" equation, i.e. of (2.10) with vanishing "external, symmetry breaking control parameter" y:

Theorem 2.1 *There exist neighbourhoods $V \subseteq X$ of $\mathcal{O}(x_0)$ and $W_j \subseteq \Lambda_j$ of λ_{0j} and C^k–maps $\hat{x}_0 : W_1 \to X_0$ and $\hat{\lambda}_2 : W_1 \to \Lambda_2$ with $\hat{x}_0(\lambda_{01}) = x_0$ and $\hat{\lambda}_2(\lambda_{01}) = \lambda_{02}$ such that it holds $F(x, \lambda_1 + \lambda_2) = 0$ with $x \in V$ and $\lambda_j \in W_j$ if and only if $\lambda_2 = \hat{\lambda}_2(\lambda_1)$ and $x = S(\gamma)\hat{x}_0(\lambda_1)$ for some $\gamma \in \mathbf{T}^n$.*

Using a more geometrical language, Theorem 2.1 can be formulated as follows:

The "unperturbed" problem $F(x, \lambda) = 0, \ x \approx \mathcal{O}(x_0), \ \lambda \approx \lambda_0$ is solvable if and only if the "internal, symmetry preserving control parameter" λ belongs to the C^k–submanifold $\mathcal{M} := \{\lambda_1 + \hat{\lambda}_2(\lambda_1) : \lambda_1 \approx \lambda_{01}\}$ (which does not depend on the choises of the subspaces Λ_1 and Λ_2). $\mathcal{M}$ has codimension n in Λ, and its tangential space in the point λ_0 is

$$T_{\lambda_0}\mathcal{M} = \{\lambda \in \Lambda : \partial_\lambda F(x_0, \lambda_0)\lambda \in \text{im}\, \partial_\lambda F(x_0, \lambda_0)\}. \tag{2.11}$$

Let us denote $\mathcal{S} := \{(\lambda_2, y) \in \Lambda_2 \times Y : \|\lambda_2\|^2 + \|y\|^2 = 1\}$ (here the symbol $\|\cdot\|$ is used for the norms in Λ and Y, respectively) and, for $\epsilon_0 > 0$, $for (\mu_0, z_0) \in \mathcal{S}$ and for neighbourhoods $W \subseteq \Lambda_1 \times \mathcal{S}$ of $(\lambda_{01}, \mu_0, z_0)$,

$$\begin{aligned} &K(\epsilon_0, \mu_0, z_0, W) := \\ &\quad := \{(\lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \epsilon z) \in \Lambda \times Y : 0 < |\epsilon| < \epsilon_0, (\lambda_1, \mu, z) \in W\}. \end{aligned} \tag{2.12}$$

Because of the applications in Sections 4 and 5 we call the sets (2.12) locking cones.

Further, for $j = 1, ..., n$ denote

$$v_j := \frac{d}{dt}\Big[T(1, ..., 1, e^{it}, 1, ..., 1)x_0)\Big]_{t=0}. \tag{2.13}$$

In (2.13), the term e^{it} stands at the j-th place in the argument of the representation T. Then $\{v_1, ..., v_n\}$ is a basis in $\ker \partial_x F(x_0, \lambda_0)$, and there exists a basis $\{v_1^*, ..., v_n^*\}$ in $\ker \partial_x F(x_0, \lambda_0)^*$ (the operator $\partial_x F(x_0, \lambda_0)^* \in \mathcal{L}(\tilde{X}^*, X^*)$ is the adjoint operator to $\partial_x F(x_0, \lambda_0)$) such that

$$\langle v_i, v_j^* \rangle = \delta_{ij}. \tag{2.14}$$

Here $\langle\cdot,\cdot\rangle : \tilde{X} \times \tilde{X}^* \to \mathbb{R}$ is the dual pairing, and δ_{ij} is the Kronecker symbol. Finally, for $j = 1, ..., n$, $(t^1, ..., t^n) \in \mathbb{R}^n$ and $(\mu, z) \in \mathcal{S}$ we set

$$H_j(t^1, ..., t^n, \mu, z) := \langle \partial_\lambda F(x_0, \lambda_0)\mu - T(e^{-it^1}, ..., e^{-it^n})z, v_j^* \rangle, \tag{2.15}$$

and by

$$\mathcal{J}(t^1, ..., t^n, \mu, z) := \frac{\partial(H_1, ..., H_n)}{\partial(t^1, ..., t^n)}(t^1, ..., t^n, \mu, z)$$

we denote the corresponding functional matrix. The system of equations $H_j(t^1, ..., t^n, \mu, z) = 0$, $j = 1, ..., n$, is the so-called reduced bifurcation equation for problem (2.10).

The following theorem describes the bifurcation from the solution orbit $\mathcal{O}(x_0)$ of solutions x to (2.10) as the control parameters λ and y move away from λ_0 and zero, respectively, their dynamic stability and their asymptotic behavior for y tending to zero.

Theorem 2.2 *Suppose $H_j(t_0^1, ..., t_0^n, \mu_0, z_0) = 0$ for $j = 1, ..., n$, and let the determinant of $\mathcal{J}(t_0^1, ..., t_0^n, \mu_0, z_0)$ be nonzero.*

Then there exist $\epsilon_0 > 0$, neighbourhoods $V \subseteq X$ of $T(e^{it_0^1}, ..., e^{it_0^n})x_0$, $W \subseteq \Lambda_1 \times \mathcal{S}$ of $(\lambda_{01}, \mu_0, z_0)$, a C^{k-1}-map $\hat{\gamma} : W \to \Gamma$ with $\hat{\gamma}(\lambda_{01}, \mu_0, z_0) = (e^{it_0^1}, ..., e^{it_0^n})$ and a C^k-map $\hat{x} : K(\epsilon_0, \mu_0, z_0, W) \to X$ such that the following is true:

(i) *It holds $F(x, \lambda) = y$ with $x \in V$ and $(\lambda, y) \in K(\epsilon_0, \mu_0, z_0, W)$ if and only if $x = \hat{x}(\lambda, y)$.*

(ii) *Let $(\lambda_1, \mu, z) \in W$ be fixed. Then $\hat{x}(\lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \epsilon z)$ tends to $S(\hat{\gamma}(\lambda_1, \mu, z))\hat{x}_0(\lambda_1)$ for $\epsilon \to 0$.*

(iii) *If $\sup\{\mathrm{Re}\,\xi : \xi \in \mathrm{spec}\ \partial_x F(x_0, \lambda_0),\ \xi \neq 0\}$ and $\max \mathrm{spec}\,\mathcal{J}(t_0^1, ..., t_0^n, \mu_0, z_0)$ are negative then, for all $(\lambda, y) \in K(\epsilon_0, \mu_0, z_0, W)$, the solution $x = \hat{x}(\lambda, y)$ is linearly stable, i.e. $\sup\{\mathrm{Re}\,\xi : \xi \in \mathrm{spec}\ \partial_x F(\hat{x}(\lambda, y), \lambda)\}$ is negative, too. If $\max \mathrm{spec}\,\mathcal{J}(t_0^1, ..., t_0^n, \mu_0, z_0)$ is positive then, for all $(\lambda, y) \in K(\epsilon_0, \mu_0, z_0, W)$, the solution $x = \hat{x}(\lambda, y)$ is linearly unstable, i.e. $\sup\{\mathrm{Re}\,\xi : \xi \in \mathrm{spec}\ \partial_x F(\hat{x}(\lambda, y), \lambda)\}$ is positive, too.*

Theorem 2.2 implies a criterion for a given subspace Λ_* of the space Λ of all internal, symmetry preserving control parameters to have the following property: For each in a certain sense nondegenerate external, symmetry breaking control parameter y near zero, it is possible to adjust λ near λ_0, by variing the components in Λ_* only, such that (2.10) gets solvable. More precise, it holds

Corollary 2.3 *Let Λ_* be a closed subspace in Λ such that the linear map*

$$\lambda_* \in \Lambda_* \longmapsto \left[\langle \partial_\lambda F(x_0, \lambda_0)\lambda_*, v_j^* \rangle\right]_{j=1}^n \in \mathbb{R}^n$$

is surjective. Then, for each $y \in Y$ near zero such that the matrix

$$\left[\partial_{t^j}\langle T(e^{-it^1},...,e^{-it^n})y, v_l^*\rangle\right]_{j,l=1}^n$$

is nonsingular in at least one point $(t^1,...,t^n) = (t_0^1,...,t_0^n) \in \mathbb{R}^n$, there exist $\lambda_* \in \Lambda_*$ near zero and $x \in X$ near $\mathcal{O}(x_0)$ such that $F(x, \lambda_0 + \lambda_*) = y$ and that $\partial_x F(x, \lambda_0 + \lambda_*)$ is an isomorphism from X onto $\tilde{X}$.

Again, using a more geometrical language, Corollary 2.3 can be formulated as follows: Let Λ_* be a closed subspace in Λ which is transversal to the subspace (2.11). Then, for each $y \in Y$ near zero such that $T_{\tilde{y}}\mathcal{O}(y)$ is transversal to $\operatorname{im}\partial_x F(x_0, \lambda_0)$ in $\tilde{X}$ for at least one $\tilde{y} \in \mathcal{O}(y)$, there exist a $\lambda_* \in \Lambda_*$ near zero and a regular solution $x \in X$ near $\mathcal{O}(x_0)$ to $F(x, \lambda_0 + \lambda_*) = y$.

Theorem 2.2 describes solution families of the problem (2.10) which are smoothly parametrized by the control parameter (λ, y) belonging to the open subset $K(\epsilon_0, \mu_0, z_0, W)$ of $\Lambda \times Y$. But Theorem 2.2 does not state any assertion about the questions whether or not these families have a smooth continuation outside of $K(\epsilon_0, \mu_0, z_0, W)$ (with the exception of the assertion of the impossibility of continuous continuation into the points $(\lambda, y) = (\lambda_1 + \hat{\lambda}_2(\lambda_1), 0)$, cf. [41, Remark 5.5]), whether or not there exists a maximal domain of definition of such a continuation and how behaves the solution x if (λ, y) tends to the boundary of such a maximal domain of continuation.

In order to answer these questions in the case $n = 1$, we use the notation (similar to (2.13) and (2.14))

$$v := \frac{d}{dt}\left[T(e^{it})x_0\right]_{t=0}, \quad v^* \in X^* : \ \partial_x F(x_0, \lambda_0)^* v^* = 0, \ \langle v, v^* \rangle = 1. \tag{2.16}$$

Further, (2.4) implies that in this case there exists a $\lambda_* \in \Lambda_2$ such that

$$\langle \partial_\lambda F(x_0, \lambda_0)\lambda_*, v^* \rangle = 1, \tag{2.17}$$

and we denote for $z \in Y$

$$\begin{aligned} \mu_+(z) &:= \max\{\langle T(e^{it})z, v^* \rangle : \ t \in \mathbb{R}\} \\ \mu_-(z) &:= \min\{\langle T(e^{it})z, v^* \rangle : \ t \in \mathbb{R}\}. \end{aligned} \tag{2.18}$$

Theorem 2.4 *Let $z_0 \in Y$ be such that the function $t \in \mathbb{R} \mapsto \langle\, T(e^{-it})z_0, v^* \rangle \in \mathbb{R}$ has exactly two critical points in $[0, 2\pi)$ and that booth these critical points are nondegenerate.*

Then there exist $\epsilon_0 > 0$, neighbourhoods $V \subseteq X$ of $\mathcal{O}(x_0)$, $W_1 \subseteq \Lambda_1$ of λ_{01}, $W_2 \subseteq \mathbb{R}$ of zero and $W \subseteq Y$ of z_0 and C^{k-1}-maps ν_+ and ν_- from $(-\epsilon_0, \epsilon_0) \times W_1 \times W$ into $\mathbb{R}$ such that

$$\nu_\pm(\epsilon, \lambda_1, z) = \epsilon\Big[\mu_\pm(z) + O(|\epsilon| + \|\lambda_1 - \lambda_{01}\|)\Big] \text{ for } |\epsilon| + \|\lambda_1 - \lambda_{01}\| \to 0 \qquad (2.19)$$

uniformly for $z \in W$, and that for all $\epsilon \in (-\epsilon_0, \epsilon_0)$, $\lambda_1 \in W_1$, $\nu \in W_2$ and $z \in W$ the following holds:

(i) *For all $\nu \in (\nu_-(\epsilon, \lambda_1, z), \nu_+(\epsilon, \lambda_1, z))$ there exist exactly two solutions $x \in V$ of the equation*

$$F(x, \lambda_1 + \widehat{\lambda}_2(\lambda_1) + \nu\lambda_*) = \epsilon z, \qquad (2.20)$$

one is linearly stable, the other is linearly unstable. These solutions depend C^k-smoothly on ϵ, λ_1, ν and z, and for $|\nu - \nu_+(\epsilon, \lambda_1, z)| \to 0$ or $|\nu - \nu_-(\epsilon, \lambda_1, z)| \to 0$ they coalesce (saddle node bifurcation).

(ii) *For $\nu \notin (\nu_-(\epsilon, \lambda_1, z), \nu_+(\epsilon, \lambda_1, z))$ there do not exist solutions $x \in V$ to (2.20).*

Remark 2.5 A similar to Theorem 2.4, but more complicated result holds if one assumes that the map $t \in \mathbb{R} \mapsto \langle\, T(e^{-it})z_0, v^*\rangle \in \mathbb{R}$ has exactly $2l$ (with $l \in \mathbb{N}$) critical points in $[0, 2\pi)$ and that all these critical points are nondegenerate.

In that case there exist $\epsilon_0 > 0$, neighbourhoods $V \subseteq X$ of $\mathcal{O}(x_0)$, $W_1 \subseteq \Lambda_1$ of λ_{01}, $W_2 \subseteq \mathbb{R}$ of zero and $W \subseteq Y$ of z_0 and C^{k-1}-maps ν_j (for $j = 1, ..., 2l$) from $(-\epsilon_0, \epsilon_0) \times W_1 \times W$ into $\mathbb{R}$ such that for $|\epsilon| + \|\lambda_1 - \lambda_{01}\| \to 0$

$$\begin{aligned} \nu_1(\epsilon, \lambda_1, z) &= \epsilon[\max\{\langle T(e^{-it})z, v^*\rangle : \gamma \in \mathbb{R}\} + O(|\epsilon| + \|\lambda_1 - \lambda_{01}\|)], \\ \nu_{2l}(\epsilon, \lambda_1, z) &= \epsilon[\min\{\langle T(e^{-it})z, v^*\rangle : \gamma \in \mathbb{R}\} + O(|\epsilon| + \|\lambda_1 - \lambda_{01}\|)] \end{aligned}$$

and $\hat{\nu}_j(\epsilon, \lambda_1, z) < \hat{\nu}_{j'}(\epsilon, \lambda_1, z)$ for $j > j'$ and $\epsilon > 0$, and that the solution behavior of (2.20) with $\epsilon \approx 0$, $\lambda_1 \approx \lambda_{01}$, $\nu \approx 0$ and $z \approx z_0$ can be described in the following way:

For $\nu \in (\nu_{2l}(\epsilon, \lambda_1, z), \nu_1(\epsilon, \lambda_1, z))$ there exist at least two (but a finite number of) solutions $x \approx \mathcal{O}(x_0)$ to (2.20). If the control parameter $(\epsilon, \lambda_1, \nu, z)$ intersects one of the hypersurfaces $\nu = \nu_j(\epsilon, \lambda_1, z)$, then the number of solutions $x \approx \mathcal{O}(x_0)$ changes generically by two (saddle node bifurcations). If $(\epsilon, \lambda_1, \nu, z)$ does not belong to one of these hypersurfaces, then the number of solutions $x \approx \mathcal{O}(x_0)$ is even, half of them are linearly stable, the other's are linearly unstable (for related results see [30, Theorem 1.1], [49, Theorem 8.5.6] and [14, Theorem 11.5.1]).

For $\nu \notin (\nu_-(\epsilon, \lambda_1, z), \nu_+(\epsilon, \lambda_1, z))$ there do not exist solutions $x \in V$ to (2.20).

Remark 2.6 Let the assumptions of Theorem 2.4 be satisfied with $X = \tilde{X} = \mathbb{R}^m$, and consider the ordinary differential equation

$$\dot{x} = F(x, \lambda) - y. \tag{2.21}$$

Then the group orbit $\mathcal{O}(x_0)$ is an attracting normally hyperbolic invariant manifold for (2.21) (cf. e.g., [35]). Hence, for $\lambda \approx \lambda_0$ and $y \approx 0$ there exists an attracting normally hyperbolic invariant manifold $M(\lambda, y)$ for (2.21) near $\mathcal{O}(x_0)$ (cf., e.g., [43, 55]). The manifold $M(\lambda, y)$ is diffeomorphic to $\mathcal{O}(x_0)$ and, hence, to $\mathbf{S}^1$. All solutions $x(t)$ of (2.21), which stay near $\mathcal{O}(x_0)$ for all times, move on $M(\lambda, y)$.

Let us consider the dynamics of (2.21) with $\lambda = \lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\lambda_*$, $y = \epsilon z$, (for small $\epsilon > 0$) in more detail.

If $\nu \in (\nu_-(\epsilon, \lambda_1, z), \nu_+(\epsilon, \lambda_1, z))$, then the two stationary solutions to (2.21), described by Theorem 2.4, lie on $M(\lambda, y)$. Hence, they are connected by two heteroclinic orbits. One of these stationary solutions is asymptotically stable, the other is unstable. For $\nu \downarrow \nu_-(\epsilon, \lambda_1, z)$ or $\nu \uparrow \nu_+(\epsilon, \lambda_1, z)$ they coalesce in a nonhyperbolic stationary solution to (2.21) (a saddle node), one of the heteroclinic orbits disappears, and the other changes into a homoclinic orbit from the saddle node.

If $\nu \notin (\nu_-(\epsilon, \lambda_1, z), \nu_+(\epsilon, \lambda_1, z))$, then there do not exist stationary solutions to (2.21) on $M(\lambda, y)$. Hence, $M(\lambda, y)$ is an attracting periodic orbit. For $\nu \uparrow \nu_-(\lambda_1, \epsilon, z)$ or $\nu \downarrow \nu_+(\lambda_1, \epsilon, z)$ this periodic orbit changes into the homoclinic orbit from the saddle node. Especially, its period tends to infinity.

The codimension one bifurcation which occures for $\nu = \nu_\pm(\lambda_1, \epsilon, z)$ is well described. In [2, Chapter 21] it is called "blue loop" and in [3, Chapter 33] "birth of a cycle from a homoclinic orbit of a saddle node" (see also [14, Chapter 10.4]).

Now, consider the case of vanishing symmetry breaking parameter, i.e. the $\mathbf{S}^1$-equivariant differential equation

$$\dot{x} = F(x, \lambda). \tag{2.22}$$

Because of Theorem 2.1, for $\lambda \in \mathcal{M}$ we have $M(\lambda, 0) = \mathcal{O}(\hat{x}_0(\lambda))$, i.e. the invariant manifold $M(\lambda, 0)$ for (2.22) consists of stationary solutions only.

For $\lambda \notin \mathcal{M}$, $M(\lambda, 0)$ is an attracting periodic orbit for (2.22) and simultaneously a group orbit. Hence, it is the orbit of a rotating wave solution of the type

$$x(t) = S(e^{i\alpha(\lambda)t})x_*(\lambda). \tag{2.23}$$

For λ on opposite sides of the hypersurface $\mathcal{M}$, the frequency $\alpha(\lambda)$ of the rotating wave (2.23) has opposite signs, i.e. the solution (2.23) rotates in opposite directions. Moreover, it holds

$$\alpha(\lambda) = O(\text{dist}(\lambda, \mathcal{M})) \text{ for } \text{dist}(\lambda, \mathcal{M}) \longrightarrow 0,$$

i.e. the period of the rotating wave solution (2.23) tends to infinity if the distance of the control parameter λ to $\mathcal{M}$ tends to zero ("freezing phenomenon", cf., e.g., [21]).

3 Notation and Assumptions

Throughout in in the following $k \geq 2$, $m \geq 2$ and n are natural numbers, $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^m$, $\mathbb{M}_m$ is the space of all real $m \times m$ -matrices, and $S : \mathbb{R} \to \mathbb{M}_m$ is a C^k-map such that $S(0) = I$, $S(\gamma + \delta) = S(\gamma)S(\delta)$, $S(\gamma + 2\pi) = S(\gamma)$ and $\langle S(\gamma)\xi, S(\gamma)\eta \rangle = \langle \xi, \eta \rangle$ for all $\gamma, \delta \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^m$. Further, $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is a C^k-map, and we assume

(I) $f(S(\gamma)\xi, \lambda) = S(\gamma)f(\xi, \lambda)$ for all $\gamma \in \mathbb{R}$, $\xi \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}^n$.

In other words, $e^{i\gamma} \in \mathbf{S}^1 \mapsto S(\gamma) \in \mathbb{M}_m$ is a C^k-smooth unitary $\mathbf{S}^1$-representation on $\mathbb{R}^m$, and the vector field $f(\cdot, \lambda)$ is equivariant with respect to this representation. Further, we set

$$S_0 := \frac{d}{d\gamma}[S(\gamma)]_{\gamma=0}. \tag{3.1}$$

Then, obviously, it holds

$$S_0^T = -S_0, \tag{3.2}$$

where S_0^T is the transposed to S_0 matrix.

The symbol $\|\cdot\|$ will be used for the Euclidean norms in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. By $C_{2\pi}$ resp. $C_{2\pi}^l$ (for $l \in \mathbb{N}$) we denote the Banach spaces of all 2π–periodic maps $x : \mathbb{R} \to \mathbb{R}^m$ that are continuous resp. C^l–smooth, equipped with the usual norms $\max\{\|x(t)\| : t \in \mathbb{R}\}$ resp.

$$\|x\|_l := \max\{\, \|x^{(j)}(t)\| : t \in \mathbb{R},\, j = 0, 1, \ldots, l\}$$

(here $x^{(j)}$ is the j-th derivative of the map x).

4 Forced Frequency Locking of Rotating Wave Solutions

In this section we consider the differential equation

$$\dot{\xi}(t) = f(\xi(t), \lambda) - S(\alpha t)y. \tag{4.1}$$

In (4.1), $y \in \mathbb{R}^m$ is small, α is real, and we regard (4.1) as an autonomous parameter depending $\mathbf{S}^1$–equivariant differential equation with a small perturbation $S(\alpha t)y$, which breaks the autonomy and the equivariance and which is of a quite special $\frac{2\pi}{\alpha}$–periodic type.

Let us introduce new "rotating" variables

$$x(t) := S(-\alpha t)\xi(t). \tag{4.2}$$

Then we get (4.1) in the equivalent form (cf. (3.1) and (3.2))

$$\dot{x}(t) = f(x(t), \lambda) - \alpha S_0 x(t) - y. \tag{4.3}$$

Thus, (4.2) transforms (4.1) into an autonomous $\mathbf{S}^1$–equivariant differential equation with a small time-independent, but symmetry breaking perturbation y.

The aim of this section is to apply the results of Section 2 to equation (4.3) and, after that, to translate the results via (4.2) into results for (4.1).

Stationary solutions $x \in \mathbb{R}^m$ to (4.3) correspond to periodic solutions

$$\xi(t) = S(\alpha t)x \tag{4.4}$$

to (4.1). Periodic solutions of the special type (4.4) (i.e. which move along group orbits) are usually called rotating waves (cf., e.g., [39, 42, 13]) or relative equilibria ([22, 34]), if the equation is autonomous and equivariant. For non-autonomous, non-equivariant equations like (4.1) this notation is somewhat unusual. Nevertheless we will use it also for such equations in order to indicate that (4.4) is a periodic solution of a special type.

Periodic solutions $x(t) \in \mathbb{R}^m$ to (4.3) with frequency $\beta > 0$ correspond via (4.2) to quasiperiodic solutions

$$\xi(t) = S(\alpha t)x(t),\ x(t + \frac{2\pi}{\beta}) = x(t) \tag{4.5}$$

to (4.1). Following RAND [39] we call solutions of the type (4.5) modulated wave solutions with modulation frequency β.

Remark 4.1 As long as $y \neq 0$ we regard the parameter α in (4.1) and (4.3) as a given external control parameter.

On the other hand, if one asks for solutions of the type (4.5) to (4.1) with $y = 0$, then one has to regard α and β as an internal state parameters which depend on λ, in general. Moreover, if $\xi(t)$ is of type (4.5), then neither α nor $x(t)$ are uniquely determined by $\xi(t)$, in general (cf. [39] and Remark 5.1 below). But, if $\xi(t)$ is not a rotating wave, then the modulation frequency β is uniquely determined by $\xi(t)$ (by the claim that $\frac{2\pi}{\beta}$ is the infimum of all $s > 0$ such that $\xi(s)$ belongs to the group orbit of $\xi(0)$).

There is an evident correspondence between stability properties of corresponding solutions to (4.1) and (4.3):

A stationary solution $x \in \mathbb{R}^n$ to (4.3) is asymptotically stable (in the usual sense, cf. [27, Definition 4.1]) iff the corresponding rotating wave solution (4.4) to (4.1) is asymptotically stable.

A periodic solution $x(t)$ to (4.3) is asymptotically orbitally stable with asymptotic phase (in the sense of [27, Definition 4.2]) iff the corresponding modulated wave solution (4.5) to (4.1) satisfies the following condition:

For all $\epsilon > 0$ there exits a $\delta > 0$ such that for all solutions $\eta(t)$ to (4.1) with $\|\eta(t) - \xi(t)\| \leq \delta$ we have that $\inf\{\|\eta(t) - S(\alpha(t-s))\xi(t)\| : s \in \mathbb{R}\} < \epsilon$ for all $t \geq 0$ and that there exits a $t_0 \in \mathbb{R}$ such that $\|\eta(t) - S(\alpha t_0)\xi(t-t_0)\| \to 0$ for $t \to \infty$. (4.6)

Of course, a non-stationary periodic solution $x(t)$ to (4.3) cannot be asymptotically stable because (4.3) is an autonomous equation. Analogously, the corresponding modulated wave solution (4.5) to (4.1) cannot be asymptotically stable (if it is not a rotationg wave), because if $\xi(t)$ is a solution to (4.1), then for all $t_0 \in \mathbb{R}$ $S(\alpha t_0)\xi(t-t_0)$ is a solution to (4.1), too.

If $y = 0$, then equation (4.1) is $\mathbf{S}^1$–equivariant, and, hence, its solutions cannot be asymptotically stable, in general. In that case the following notation, which is due to RENARDY [42], is useful. We will formulate this property in terms of equation (4.1) (with fixed λ), but we will use it for equation (4.3) as well.

Definition 4.2 A solution $\xi(t)$ to (4.1), which is defined for all $t \in \mathbb{R}$, is called asymptotically $\mathbf{S}^1$–orbitally stable with asymptotic phase if for each $\epsilon > 0$ there exists a $\delta > 0$ such that for all solutions $\eta(t)$ to (4.1) with $\inf\{\|\eta(0) - S(\gamma)\xi(s)\| : \gamma, s \in \mathbb{R}\} < \delta$ we have that $\eta(t)$ is defined for all $t \in \mathbb{R}$, that $\inf\{\|\eta(t) - S(\gamma)\xi(s)\| : \gamma, s \in \mathbb{R}\} < \epsilon$ for all $t \geq 0$ and that there exist real γ_0 and t_0 such that $\|\eta(t) - S(\gamma_0)\xi(t+t_0)\| \to 0$ for $t \to \infty$.

Remark 4.3 For stationary solutions Definition 4.2 coincides with the notation introduced in [24, Section VIII.4], cf. also [41, Remark 6.2].

For rotating wave solutions the time orbit is equal to the group orbit. Therefore, the orbital stability which relates to the time orbit [27, Definition 4.2] coincides for such solutions with Definition 4.2, i.e. with the orbital stability which relates to the group orbit.

In this section $x_0 \in \mathbb{R}^m$, $\lambda_0 \in \mathbb{R}^n$ and $\alpha_0 > 0$ are fixed such that

$$\xi_0(t) := S(\alpha_0 t)x_0, \tag{4.7}$$

is a non-stationary rotating wave solution to equation (4.1) with $\lambda = \lambda_0$ and $\alpha = \alpha_0$. Obviously, this is equivalent to

(II) $\quad \alpha_0 S_0 x_0 = f(x_0, \lambda_0),\ S_0 x_0 \neq 0.$

From (I) and (II) follows that $S_0 x_0$ is an eigenvector to the eigenvalue zero of the matrix $\partial_x f(x_0, \lambda_0) - \alpha_0 S_0$. We suppose the following condition to be satisfied:

(III) Zero is a simple eigenvalue of $\partial_x f(x_0, \lambda_0) - \alpha_0 S_0$, and the imaginary parts of all other eigenvalues are negative.

Assumption (III) yields that the stationary solution x_0 to equation (4.3) with $\lambda = \lambda_0$ and $y = 0$ is asymptotically $\mathbf{S}^1$–orbitally stable with asymptotic phase (cf. [41, Remark 6.2]) and, hence, that the rotating wave solution $\xi_0(t)$ (cf. (4.7)) to (4.1) with $\lambda = \lambda_0$ and $y = 0$ is asymptotically $\mathbf{S}^1$–orbitally stable with asymptotic phase, too. Hence, the purpose of this section is to describe the behavior of an asymptotically $\mathbf{S}^1$–orbitally stable rotating wave solution of an autonomous $\mathbf{S}^1$–equivariant differential equation under a small perturbation of the "rotating wave type" $S(\alpha t)y$.

This problem seems to be the essential qualitative part of the mathematical modeling of pulse generation by continuous light injection into a continuous wave operated laser (cf. [36, 46, 8, 5, 56]). In these applications $S(\alpha t)y$ describes the external injected light (with frequency α and time-indepentent intensity $\|S(\alpha t)y\| = \|y\|$), the rotating wave solution (4.7) is the "stationary" state of the laser without injection (with frequency α_0 and time-independent intensity $\|S(\alpha_0 t)x_0\| = \|x_0\|$), and λ describes the internal laser parameters (laser currents, geometric and material parameters, facet reflectivities).

For related results (where $f(\cdot, \lambda_0)$ is linear, and $f(\cdot, \lambda)$ is discontinuous in certain arguments) with applications in nonlinear rotordynamics see [51].

In the following we will apply the results of Section 2 to equation (4.3). Hence, we introduce an appropriate setting which satisfies the assumptions of Section 2.

We set

$$\begin{aligned} &X = \tilde{X} = Y = \mathbb{R}^m,\ \Lambda_1 = \mathbb{R}^n \text{ (the } \lambda\text{-space)},\ \Lambda_2 = \mathbb{R} \text{ (the } \alpha\text{-space)}, \\ &F(x, \lambda, \alpha) = f(x, \lambda) - \alpha S_0 x. \end{aligned} \tag{4.8}$$

Then, because of assumption (I), $F(\cdot, \lambda, \alpha)$ is equivariant with respect to the $\mathbf{S}^1$-representation S for all λ and α, and assumptions (2.5), (2.7) and (2.8) are satisfied (with

$S = T$). Further, (2.1), (2.3) and (2.9) (with λ_0 replaced by (λ_0, α_0)) are satisfied because of assumption (II). Finally, (2.4) holds because $\partial_\alpha F(x_0, \lambda_0, \alpha_0) = -S_0 x_0$ spans $\ker \partial_x F(x_0, \lambda_0, \alpha_0)$. The vector $S_0 x_0$ plays the role of the vector v in (2.16), and $v^* \in \mathbb{R}^m$ is defined by

$$[\partial_x f(x_0, \lambda_0)^T + \alpha_0 S_0] v^* = 0, \quad \langle S_0 x_0, v^* \rangle = 1. \tag{4.9}$$

(cf. (2.16) and (3.2)).

Now we apply Theorem 2.1 in order to describe the solution behavior of the unperturbed equation (i.e. of equation (4.1) with $y = 0$) near the given rotating wave solution (4.7).

Theorem 4.4 *Suppose* (I)–(III). *Then there exist a neighbourhood $W \subset \mathbb{R}^n$ of λ_0 and C^k-maps $\hat{u} : W \to \mathbb{R}^m$ and $\hat{\alpha} : W \to \mathbb{R}$ with $\hat{u}(\lambda_0) = 0$, $\hat{\alpha}(\lambda_0) = \alpha_0$ and $\langle \hat{u}(\lambda), v^* \rangle = 0$ for all $\lambda \in W$ such that*

$$\xi(t) = S(\hat{\alpha}(\lambda)t)(x_0 + \hat{u}(\lambda)) \tag{4.10}$$

is an asymptotically S^1–orbitally stable rotating wave solution with asymptotic phase to equation (4.1) *with $y = 0$.*

Remark 4.5 In fact, Theorem 4.4 follows immediately from the classical result about autonomous perturbations of hyperbolic periodic orbits of autonomous differential equations (cf., e.g., [26, Theorem VI.4.1]). Indeed, the assumptions (I)–(III) provide that the solution (4.7) is hyperbolic. Therefore, the corresponding orbit persists as a hyperbolic periodic orbit and, hence, as a rotating wave solution for (4.1) with $\lambda \approx \lambda_0$ and $y = 0$, and the period depends smoothly on λ.

Let us introduce the set $\mathcal{S}$ of "directions" in the control parameter space $\mathbb{R} \times \mathbb{R}^m$,

$$\mathcal{S} := \{(\alpha, y) \in \mathbb{R} \times \mathbb{R}^m : |\alpha|^2 + \|y\|^2 = 1\},$$

and the locking cones (for $\epsilon_0 > 0$, $(\mu_0, z_0) \in \mathcal{S}$ and neighbourhoods $W \subset \mathbb{R}^n \times \mathcal{S}$ of (λ_0, μ_0, z_0), cf. (2.12))

$$K(\epsilon_0, \mu_0, z_0, W) := \{(\lambda, \hat{\alpha}(\lambda) + \epsilon\mu, \epsilon z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m : 0 < |\epsilon| < \epsilon_0, (\lambda, \mu, z) \in W\}.$$

The map H_1 (cf. (2.15)), which defines the reduced bifurcation equation, in the setting (4.8) has the following form (cf. (2.15) and (4.9)):

$$H_1(\gamma, \mu, z) = -\mu - \langle S(-\gamma)z, v^* \rangle \quad \text{for } \gamma \in \mathbb{R} \text{ and } (\mu, z) \in \mathcal{S}. \tag{4.11}$$

Hence, Theorems 2.2 implies the following description of the bifurcation of rotating wave solutions to (4.1) (from certain phase shifts of the rotating wave solution (4.7)), their dynamic stability and their asymptotic behavior for y tending to zero:

Theorem 4.6 *Suppose* (I) – (III), *and let* $\gamma_0 \in \mathbb{R}$ *and* $(\mu_0, z_0) \in \mathcal{S}$ *be such that* $\mu_0 = -\langle S(-\gamma_0)z_0, v^*\rangle$, *and* $c := \langle S_0 S(-\gamma_0)z_0, v^*\rangle \neq 0$.

Then there exist an $\epsilon_0 > 0$, *a neighbourhood* $W \subset \mathbb{R}^n \times \mathcal{S}$ *of* (λ_0, μ_0, z_0), *a* C^{k-1}*–map* $\hat{\gamma} : W \to \mathbb{R}$ *with* $\hat{\gamma}(\lambda_0, \mu_0, z_0) = \gamma_0$ *and a* C^k*–map* $\hat{x} : K(\epsilon_0, \mu_0, z_0, W) \to \mathbb{R}^m$ *such that the following holds:*

(i) *For* $(\lambda, \alpha, y) \in K(\epsilon_0, \mu_0, z_0, W)$ *is* $\hat{\xi}(t, \lambda, \alpha, y) := S(\alpha t)\hat{x}(\lambda, \alpha, y)$ *a rotating wave solution to* (4.1). *It is asymptotically stable (resp. unstable) if* $c < 0$ *(resp.* $c > 0$*).*

(ii) *Let* $(\lambda, \mu, z) \in W$ *be fixed. Then* $\hat{x}(\lambda, \hat{\alpha}(\lambda) + \epsilon\mu, \epsilon z) \to S(\hat{\gamma}(\lambda, \mu, z))(x_0 + \hat{u}(\lambda))$ *for* $\epsilon \to 0$, *and, hence, the rotating wave solution* $\hat{\xi}(t, \lambda, \hat{\alpha} + \epsilon\mu, \epsilon z)$ *to* (4.1) *tends to the phase shift* $S(\hat{\alpha}(\lambda)t + \hat{\gamma}(\lambda, \mu, z)(x_0 + \hat{u}(\lambda))$ *of the rotating wave solution* (4.10) *to* (4.1) *with* $y = 0$.

In order to apply Theorem 2.4 we define for $z \in \mathbb{R}^m$ (cf. (2.18))

$$\begin{aligned} \mu_+(z) &:= \max\{\langle S(\gamma)z, v^*\rangle : \gamma \in \mathbb{R}\}, \\ \mu_-(z) &:= \min\{\langle S(\gamma)z, v^*\rangle : \gamma \in \mathbb{R}\}. \end{aligned} \tag{4.12}$$

The following theorem describes, on the one hand, the control parameters $\lambda \approx \lambda_0$ and $y \approx 0$ such that exactly one phase shift of (4.7) persists as a stable rotating wave solution to (4.1), and, on the other hand, the creation of a one parameter family of stable (in the sense of (4.6)) modulated wave solutions near the orbit of (4.7):

Theorem 4.7 *Suppose* (I) – (III), *and let* $z_0 \in \mathbb{R}^m$ *be such that there exist exactly two solutions* γ_1 *and* γ_2 *of equation* $\langle S(\gamma)z_0, S_0 v^*\rangle = 0$ *in* $[0, 2\pi)$. *Further, suppose* $\langle S(\gamma_j)z_0, S_0^2 v^*\rangle \neq 0$ *for* $j = 1, 2$.

Then there exist $\epsilon_0 > 0$, *neighbourhoods* $W_1 \subset \mathbb{R}^n$ *of* λ_0, $W_2 \subset \mathbb{R}$ *of* α_0 *and* $W \subset \mathbb{R}^m$ *of* z_0 *and* C^{k-1}*-maps* α_+ *and* α_- *from* $(-\epsilon_0, \epsilon_0) \times W_1 \times W \to \mathbb{R}$ *such that*

$$\alpha_\pm(\epsilon, \lambda, z) = \hat{\alpha}(\lambda) + \epsilon\Big[\mu_\pm(z) + O(|\epsilon| + \|\lambda - \lambda_0\|)\Big] \text{ for } |\epsilon| + \|\lambda - \lambda_0\| \to 0 \tag{4.13}$$

and that for all $\epsilon \in (0, \epsilon_0)$, $\lambda \in W_1$, $\alpha \in W_2$ *and* $z \in W$ *the following holds:*

(i) *If* $\alpha \in (\alpha_-(\epsilon, \lambda, z), \alpha_+(\epsilon, \lambda, z))$ *then there exist two rotating wave solutions near* $\mathcal{O}(x_0)$ *with frequency* α *to equation* (4.1) *with* $y = \epsilon z$. *One is asymptotically stable, the other is unstable. They depend* C^k*-smoothly on* λ, α *and* $y = \epsilon z$, *and for* $|\alpha - \alpha_\pm(\epsilon, \lambda, z)| \to$

0 *they coalesce in a non-hyperbolic rotating wave solution (saddle node bifurcation of rotating waves).*

(ii) *If* $\alpha \notin (\alpha_-(\epsilon, \lambda, z), \alpha_+(\epsilon, \lambda, z))$ *then there exists a family of modulated wave solutions*

$$S(\alpha t)\hat{x}(t + \varphi, \lambda, \alpha, \epsilon, z) \tag{4.14}$$

near $\mathcal{O}(x_0)$ *to* (4.1) *with* $y = \epsilon z$. *In* (4.14), $\varphi \in \mathbb{R}$ *is the family parameter, and* $\hat{x}(t, \lambda, \alpha, \epsilon, z) \in \mathbb{R}^m$ *depends* C^k*-smoothly on its arguments and is periodic with respect to* t. *All the solutions* (4.14) *satisfiy the stability property* (4.6). *For* $|\alpha - \alpha_\pm(\epsilon, \lambda, z)| \to 0$ *all the modulated wave solutions* (4.14) *tend to the non-hyperbolic rotating wave solution from* (i), *and the modulation frequency of* (4.14) *tends to infinity. Moreover,*

$$\hat{x}(t, \lambda, \alpha, \epsilon, z) \to S((\hat{\alpha}(\lambda) - \alpha)t)(x_0 + \hat{u}(\lambda)) \ \text{for } \epsilon \to 0. \tag{4.15}$$

Proof The proof is a straightforward application of the results of Remark 2.6 to equations (4.3) and (by translation via (4.2)) (4.1). Especially, because of (4.9) the role of λ_* in (2.17) plays the number -1.

Let us proceed the proof of (4.15), only. The function $\hat{x}(t, \lambda, \alpha, \epsilon, z)$ is a periodic solution to (4.3) with $y = \epsilon z$, which tends for $\epsilon \to 0$ to a rotating wave solution $S(\beta t)x_*$ to (4.3) with $\epsilon = 0$. Hence,

$$f(x_*, \lambda) - (\alpha + \beta)S_0 x_* = 0.$$

Therefore, Theorem 2.1 yields $\alpha + \beta = \hat{\alpha}(\lambda)$, $x_* = x_0 + \hat{u}(\lambda)$ (cf. Theorem 4.4). ■

Remark 4.8 Let us weaken the assumptions of Theorem 4.7 supposing that the equation $\langle S(\gamma)z_0, S_0 v^* \rangle = 0$ has regular solutions γ, only (and not, in addition, the the number of such solution is two). Then the assertions of Theorem 4.7 have to be changed as it is described in Remark 2.5. Especially, assertion (ii) of Theorem 4.7 remains almost unchanged. The only possible (but nongeneric) modification is that, for $|\alpha - \alpha_\pm(\epsilon, \lambda, z)| \to 0$, the modulated wave solution (4.14) creates several non-hyperbolic rotating wave solutions.

Now, let us apply Corollary 2.3 to equation (4.1).

Remark that up to now we did not suppose any assumption concerning the λ-dependence of the vector field $f(\cdot, \lambda)$. For example, up to here $f(\cdot, \lambda)$ could be λ-independent. Now, to the contrary, we suppose that the gradient in $\lambda = \lambda_0$ of the function $\lambda \mapsto \langle f(x_0, \lambda), v^* \rangle$ is nonzero.

In applications, the parameters in the subspace Λ_*, considered below, are distinguished by the property that it is much easier to vary them (by reasons of the technology of the real system which is modeled by equation (4.1)) then other internal, symmetry preserving parameters. Often one distinguishes "bifurcation parameters" (which are candidates for Λ_*) from "system parameters". In laser modeling, for example, the parameters in Λ_* are the laser currents which are much easier to vary then the other laser parameters (as geometric and material parameters or facet reflectivities).

Corollary 4.9 *Suppose* (I) – (III), *and let* $\Lambda_* \subseteq \mathbb{R}^n$ *be a subspace such that the map* $\lambda_* \in \Lambda_* \mapsto \langle \partial_\lambda f(x_0, \lambda_0)\lambda_*, v^* \rangle \in \mathbb{R}$ *is surjective.*

Then, for each $\alpha \approx \alpha_0$ *and each small* $y \in \mathbb{R}^m$ *such that the function* $\gamma \in \mathbb{R} \mapsto \langle S(\gamma)y, S_0 v^* \rangle \in \mathbb{R}$ *has regular zeros only, the following holds: The parameter* λ *may be adjusted near* λ_0, *by varying the components in* Λ_* *only, such that equation* (4.1) *has asymptotically stable* $\frac{2\pi}{\alpha}$*-periodic rotating wave solutions (resp. modulated wave solutions, wich are stable in the sense of* (4.6)*) near the orbit* $\{\xi_0(t) \in \mathbb{R}^m : t \in \mathbb{R}\}$ *of the unperturbed rotating wave solution* (4.7).

In applications (cf., [36, 46, 8, 5]), often one is interested in modulated wave solutions to (4.1) with high modulation frequency and with large "modulation oscillation", i.e. in solutions of the type (4.5) with large β and with large

$$\max\{\|x(t)\| : t \in \mathbb{R}\} - \min\{\|x(t)\| : t \in \mathbb{R}\}. \tag{4.16}$$

Theorem 4.6 states that (in the described situation) the possibilities to come up to these demands are quite limited:

Let, for example, $\mu_+(z_0) > 0$ (cf. (4.12)). Further, let $\epsilon = \epsilon_+(\alpha, \lambda, z)$ be the solution of equation $\alpha = \alpha_+(\epsilon, \lambda, z)$ with $\epsilon \approx 0$, $\lambda \approx \lambda_0$, $\alpha \approx \hat{\alpha}(\lambda)$ and $z \approx z_0$. Then (cf. (4.13))

$$\epsilon_+(\alpha, \lambda, z) = \frac{\alpha - \hat{\alpha}(\lambda)}{\mu_+(z)} + o(|\alpha - \hat{\alpha}(\lambda)|) \text{ for } |\alpha - \hat{\alpha}(\lambda)| \to 0.$$

Finally, let $\lambda \approx \lambda_0$, $\alpha \approx \hat{\alpha}(\lambda)$, $\alpha > \hat{\alpha}(\lambda)$ and $z \approx z_0$ be fixed. Then for $\epsilon \approx 0$ the oscillation (4.16) for $\hat{x}(\cdot, \lambda, \alpha, \epsilon, z)$ is small (because of (4.15)). If one tries to increase this oscillation by increasing ϵ one has to pay for this by decreasing the frequency of $\hat{x}(\cdot, \lambda, \alpha, \epsilon, z)$, because this frequency tends to zero for $\epsilon \to \epsilon_+(\alpha, \lambda, z)$.

In other words, if one tries to get modulated wave solutions near a given rotating wave solution of an $\mathbf{S}^1$–equivariant differential equation by forcing this equation by an (unmodulated) rotating wave, then one gets the following:

For small forcings (i.e. for $0 < \epsilon \ll \epsilon_+(\alpha, \lambda, z)$) small modulation oscillations are created, but the modulation frequencies are large. The modulation frequency is near the difference between the frequency of the rotating wave solution of the unforced equation (which is $\hat{\alpha}(\lambda)$, if $S(\gamma)x_0 = x_0$ only for $\gamma \in 2\pi\mathbb{Z}$) and the frequency of the forcing (which is α, if $S(\gamma)y = y$ only for $\gamma \in 2\pi\mathbb{Z}$).

On the other hand, for large forcings (i.e. for $0 \ll \epsilon < \epsilon_+(\alpha, \lambda, z)$) large modulation oscillations occure, but the modulation frequencies are small. If ϵ tends to $\epsilon_+(\alpha, \lambda, z)$ from below then the modulation frequency tends to zero, and the modulated wave solution changes "back" into two rotating wave solutions (if z_0 satisfies the assumptions or Theorem 4.6). These rotating wave solutions are close to fixed phase shifts of the "initial" rotating wave solution (4.7) and have exactly the same frequency as the forcing. One of them is stable, the other is unstable. In this sense, frequency locking of the rotating wave solution $\xi(t) = S(e^{i\alpha_0 t})x_0$ of the unperturbed equation $\dot{\xi} = f(\xi, \lambda_0)$ with a forcing $\eta(t) = S(e^{i\alpha t})y$ of "rotating wave type" occures.

If ϵ is increased further (i.e. $\epsilon > \epsilon_+(\alpha, \lambda, z)$) then the following takes place:

In the case of $\mu_-(z) < 0$ the locked rotating wave solutions change quantitatively, only. No modulated wave solutions occure.

But in the case of $\mu_-(z) > 0$ there exists a positive solution $\epsilon = \epsilon_-(\alpha, \lambda, z)$ of equation $\alpha = \alpha_-(\epsilon, \lambda, z)$ with $\epsilon \approx 0$, $\lambda \approx \lambda_0$, $\alpha \approx \hat{\alpha}(\lambda)$, $\alpha > \hat{\alpha}(\lambda)$ and $z \approx z_0$. It holds

$$\epsilon_-(\alpha, \lambda, z) = \frac{\alpha - \hat{\alpha}(\lambda)}{\mu_-(z)} + o(|\alpha - \hat{\alpha}(\lambda)|) \text{ for } |\alpha - \hat{\alpha}(\lambda)| \to 0$$

and, hence, $\epsilon_-(\alpha, \lambda, z) > \epsilon_+(\alpha, \lambda, z)$ for $\alpha > \hat{\alpha}(\lambda)$. In this case the locking cone

$$\{(\alpha, \epsilon) : \ \alpha > \hat{\alpha}(\lambda), \ \epsilon_+(\alpha, \lambda, z) < \epsilon < \epsilon_-(\alpha, \lambda, z)\}, \ \lambda \text{ and } z \text{ fixed},$$

is lop-sided, i.e. it does not contain the axis $\alpha = \hat{\alpha}(\lambda)$. If ϵ tends to $\epsilon_-(\alpha, \lambda, z)$ from below then the two rotating wave solutions coalesce and disappear (saddle node bifurcation of rotating waves), and, again, a stable (in the sense of (4.6)) modulated wave solution occures. If ϵ tends to $\epsilon_-(\alpha, \lambda, z)$ from above then the modulation frequency of the new modulated wave solution tends to zero. Hence, if one wants to get a large modulation frequency and a large modulation oscillation, one has further to increse ϵ (as long as the local description of the solution behavior, given by Theorem 4.6, is valid).

The bifurcation scenarios in the case $\mu_+(z_0) > 0$, $\alpha < \hat{\alpha}(\lambda)$ and in the cases with $\mu_+(z_0) < 0$ may be described analogously.

Remark 4.10 From standard results on finite dimensional representations of compact Lie groups follows that the function $\gamma \in \mathbb{R} \to \langle S(\gamma)z, v^* \rangle \in \mathbb{R}$ is a trigonometrical polynom of finite order. Hence, the sums in [41, equations (7.17)–(7.20)] have a finite number of summands, only, and the corresponding criteria [41, Lemmata 7.4 and 7.5] for z_0 to satisfy the assumptions of Theorem 4.6 and for μ to belong the interval $(\mu_-(z), \mu_+(z))$ (cf. (4.12)) are quite simple.

Remark 4.11 It is well known that a small $\frac{2\pi}{\alpha}$-periodic forcing can generate the bifurcation of $\frac{2\pi q}{\alpha}$-periodic solutions (the q-subharmonic solutions with $q \in \mathbb{N}$) from a hyperbolic $\frac{2\pi}{\alpha}$-periodic solution of an autonomous ordinary differential equation (cf. e.g., [14, Section 11.2], [25, Section 4.6], [15, Chapter 7] and [9]). The forcings which generate such bifurcations may be determined by the methods of Section 2 (by scaling the time to rule $t_{new} = \alpha t_{old}$ and, after that, working in spaces of $2\pi q$-periodic vector functions instead of using "rotating" variables (4.2) and working in $\mathbb{R}^m$, cf. (4.8) and (5.9)). The corresponding reduced bifurcation equation is of the type of equation (2.7) of [14, Chapter 11].

Especially, if the unperturbed equation is $\mathbf{S}^1$-equivariant, then the hyperbolic $\frac{2\pi}{\alpha}$-periodic solution of the unperturbed equation is a rotating wave solution, and if the $\frac{2\pi}{\alpha}$-periodic forcing is of "rotating wave type" $S(\alpha t)y$ (with $y \in \mathbb{R}^m$), then it is easy to verify the following results:

For $q = 1$ the reduced bifurcation equation, obtained by working in spaces of 2π-periodic functions, is equivalent to the reduced bifurcation equation $\mu = -\langle S(-\gamma)z, v^*\rangle$ (cf. (4.11)), obtained by using the "rotating" coordinates (4.2) and by working in $\mathbb{R}^m$. Both approaches produce the same results.

But for $q > 1$ the left hand side $H_1(\gamma, \mu, z)$ of the reduced bifurcation equation (cf. (2.15)), obtained by working in spaces of $2\pi q$-periodic functions, does not depend on γ and z. Hence, Theorems 2.2 and 2.4 cannot work. Moreover, no forcing of $\frac{2\pi}{\alpha}$-periodic "rotating wave type" $S(\alpha t)y$ can satisfy the assumption of Corollary 2.3 because of

$$\int_0^{2\pi q} \langle S(t)y, S(\frac{p}{q}t)v^*\rangle \, dt = 0 \text{ for all } \ p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ with } p \neq q.$$

Thus, Theorems 2.2 and 2.4 and Corollary 2.3 do not produce results about the bifurcation of $\frac{2\pi q}{\alpha}$-periodic solutions to (4.1) from the rotating wave solution (4.7). On the other hand, such $\frac{2\pi q}{\alpha}$-periodic solutions exist (the modulated wave solutions, described above, are $\frac{2\pi q}{\alpha}$-periodic if the modulation frequency is equal to $\frac{\alpha}{q}$, and for all sufficiently large q such modutated wave solutions exist for certain control parameters $\lambda \approx \lambda_0$, $\alpha \approx \alpha_0$ and $y \approx 0$) , but they cannot be rotating wave solutions (if the forcing is nonstationary): Indeed, if $\xi(t) = S(\frac{\alpha}{q}t)x$ (with $x \in \mathbb{R}^m$) would be a solution to (4.1), then it would follow $\frac{\alpha}{q}S_0 x - f(x, \lambda) = S(\alpha(1 - \frac{1}{q})t)y$, i.e. the forcing would be stationary. Moreover, they cannot be hyperbolic (as periodic solutions).

Remark 4.12 Using [41, Remarks 3.5, 5.7, 6.3 and 7.5], the generalization of Theorems 4.5 and 4.6 to equations of the type

$$\dot{\xi}(t) = f(\xi(t), \lambda) - S(\alpha t) y(S(-\alpha t)\xi(t), \lambda) \tag{4.17}$$

is straightforward. In (4.17), the exterior, symmetry breaking control parameter y variies in the space of all maps form $\mathbb{R}^m \times \mathbb{R}^n$ into $\mathbb{R}^m$ such that all derivatives up to order k are continuous and bounded, for example (cf. [41, Remark 3.5]). The representation of the group $\mathbf{S}^1$ on this space is $y \mapsto S(\gamma)y(S(\gamma)^{-1}\cdot, \cdot)$. Hence, in Theorems 4.6 and 4.7 one has to replace the symmetry breaking parameter directions $z_0 \in \mathbb{R}^m$ and $z \in \mathbb{R}^m$ by $z_0(x_0, \lambda_0)$ and $z(x_0, \lambda_0)$, respectively.

Let us remark that, considering initial boundary value problems with dynamic boundary conditions of "rotating wave type", evolution equations of the formal type (4.17) occure in a quite natural way: Consider the evolution problem

$$\begin{aligned} \partial_t \xi(t,x) &= A(x)\xi(t,x) + f(x, \xi(t,x), \lambda), \ x \in \Omega, & (4.18)\\ \xi(t,x) &= S(\alpha t) y(x), \ x \in \partial\Omega, & (4.19) \end{aligned}$$

where $A(x)$ is a differential operator, which contains spatial derivatives only, and let $A(x)$ and $f(x, \cdot, \lambda)$ be $\mathbf{S}^1$–equivariant. Often one replaces the dynamic boundary conditions (4.19) by homogeneous one's by means of the ansatz

$$\xi(t,x) = S(\alpha t)\tilde{y}(x) + \tilde{\xi}(t,x), \tag{4.20}$$

where $\tilde{y} : \Omega \to \mathbb{R}^m$ is a suitable extension of $y : \partial\Omega \to \mathbb{R}^m$. The substitution of (4.20) into (4.18) gives for $x \in \Omega$

$$\partial_t \tilde{\xi}(t,x) = A(x)\tilde{\xi}(t,x) + S(\alpha t)[A(x)\tilde{y}(x) - \alpha_0 S_0 \tilde{y}(x) + f(x, \tilde{y}(x) + S(-\alpha t)\tilde{\xi}(t,x), \lambda)],$$

which is of the type (4.17).

5 Forced Frequency Locking of Modulated Wave Solutions

In this section we consider the differential equation

$$\dot{\xi}(\tau) = f(\xi(\tau), \lambda) - S(\alpha\tau) y(\beta\tau). \tag{5.1}$$

In (5.1), $y \in C^k_{2\pi}$ is assumed to be small (i.e. a small C^k–smooth 2π–periodic vector function), α and β are real, and we regard (5.1) as an autonomous parameter depending $\mathbf{S}^1$–equivariant differential equation with the small quasiperiodic perturbation $S(\alpha\tau)y(\beta\tau)$, which breaks the autonomy and the equivariance.

Analogously to Section 4, let us introduce new variables

$$t := \beta\tau,\ x(t) := S(-\frac{\alpha}{\beta}\tau)\xi(\tau). \tag{5.2}$$

Then we get (5.1) in the equivalent form

$$\beta\dot{x}(t) = f(x(t),\lambda) - \alpha S_0 x(t) - y(t). \tag{5.3}$$

Thus, (5.2) transforms (5.1) into an autonomous $\mathbf{S}^1$-equivariant differential equation with a small periodic, symmetry breaking perturbation $y(t)$. The aim of this section is to apply the results of Section 2 to equation (5.3) and, after that, to translate the results via (5.2) into results for (5.1).

Let $x_0 \in C^1_{2\pi}$, $\lambda_0 \in \mathbb{R}^m$, $\alpha_0 > 0$ and $\beta_0 > 0$ are fixed such that

$$\xi_0(\tau) := S(\alpha_0\tau)x_0(\beta_0\tau) \tag{5.4}$$

is a modulated wave solution to (5.1) with $\lambda = \lambda_0$ and $y = 0$. Obviously this is equivalent to

(IV) $\quad \beta_0\dot{x}_0(t) = f(x_0(t),\lambda_0) - \alpha_0 S_0 x_0(t).$

Further, we assume

(V) $\quad \dot{x}_0$ and $S_0 x_0$ are linearly independent (as maps).

It is easy to verify that (IV) and (V) imply that ξ_0 is not a rotating wave, that the invariant (with respect to the flow of equation (5.1) with $\lambda = \lambda_0$ and $y = 0$) set

$$\mathcal{T} := \{S(\gamma)x_0(t) : \gamma, t \in \mathbb{R}\} \tag{5.5}$$

is diffeomorphic to a 2-torus (and, hence, the dimension m of the phase space has to be larger than two) and that there exists a natural number l such that

$$\{\gamma \in \mathbb{R} : S(\gamma)x_0(t) = x_0(t)\} = \frac{2\pi}{l}\mathbb{Z} \text{ for all } t \in \mathbb{R} \tag{5.6}$$

(cf. [39]). Moreover, together with the modulated wave solution ξ_0 its "temporal" phase shifts $\xi(\cdot + \tau_0)$ and its "spatial" phase shifts $S(\tau_0)\xi(\cdot)$ are modulated wave solutions to equation (5.1) with $\lambda = \lambda_0$ and $y = 0$, too. Hence, we are going to describe the perturbation behavior of a two parameter family of modulated wave solutions.

Remark 5.1 The "wave frequency" α_0 in (5.4) is not uniquely determined by the modulated wave solution ξ_0, because the right hand side in (5.4) does not change if α_0 and x_0 are replaced by

$$\alpha_p := \alpha_0 + \frac{p}{l}\beta_0 \text{ and } x_p(t) := S(-\frac{p}{l}t)x_0(t), \tag{5.7}$$

where p is an arbitrary integer, and l is the "wave number" of x_0 defined by (5.6). Therefore, in what follows all the assumptions and results do not change if α_0 and x_0 are simultaneously replaced by α_p and x_p.

In applications, however, often one of the "wave frequencies" α_p is distinguished among the others. For example, if

$$\frac{1}{2\pi}\| \int_0^{2\pi} x_0(t)dt \|^2 > \int_0^{2\pi} \|x_0(t) - \frac{1}{2\pi}\int_0^{2\pi} x_0(s)ds \|^2 dt, \tag{5.8}$$

i.e. if in the Fourier expansion of x_0 the term of order zero dominates all other terms, then no x_p with $p \neq 0$ will satisfy (5.8) (with x_0 replaced by x_p), in general.

In the following we will apply Theorems 2.1, 2.2 and Corollary 2.3 to equation (5.3). Hence, we introduce an appropriate setting.

We set

$$\begin{aligned} &X = C^1_{2\pi},\ \tilde{X} = C_{2\pi},\ Y = C^k_{2\pi}, \\ &\Lambda_1 = \mathbb{R}^n \text{ (the } \lambda\text{-space)},\ \Lambda_2 = \mathbb{R}^2 \text{ (the } (\alpha,\beta)\text{-space)}, \\ &F(x,\lambda,\alpha,\beta)(t) = -\beta\dot{x}(t) + f(x(t),\lambda) - \alpha S_0 x(t). \end{aligned} \tag{5.9}$$

Obviously, $F(\cdot,\lambda,\alpha,\beta)$ is equivariant with respect to the $\mathbf{T}^2$-representation

$$T(e^{i\gamma},e^{i\delta})x := S(\gamma)x(\cdot+\delta) \text{ for } (e^{i\gamma},e^{i\delta}) \in \mathbf{T}^2 \tag{5.10}$$

for all λ, α and β. Remark that the representation T on the space $C_{2\pi}$ (as well as on $C^1_{2\pi}$ or $C^k_{2\pi}$) is not C^1-smooth, but it satisfies (2.5), (2.7) and (2.8). Moreover,the ordinary differential operator

$$\partial_x F(x_0,\lambda_0,\alpha_0,\beta_0) = -\beta_0\frac{d}{dt} + \partial_x f(x_0,\lambda_0) - \alpha_0 S_0 \tag{5.11}$$

is a Fredholm operator from $C^1_{2\pi}$ into $C_{2\pi}$ (cf., e.g., [26, Section IV.1]), therefore assumption (2.2) is satisfied.

We assume, in addition to (I), (IV) and (V), that

(VI) $\dim\{v \in C^1_{2\pi} : \beta_0\dot{v} = \partial_x f(x_0,\lambda_0)v - \alpha_0 S_0 v\} = 2$

and that there exist $v_1^*, v_2^* \in C^1_{2\pi}$ with

(VII) $\quad -\beta_0 v_j^* = \partial_x f(x_0,\lambda_0)^T v_j^* + \alpha_0 S_0 v_j^*, \;\; \int_0^{2\pi} \langle v_i(t), v_j^*(t)\rangle dt = \delta_{ij}$

for $i,j = 1,2$ and $v_1 := S_0 x_0, \; v_2 := \dot{x}_0$. In other words, we suppose that zero is a semi-simple eigenvalue of (5.11) of multiplicity two. Hence, (2.3) and (2.9) are fulfilled (with λ_0 replaced by $(\lambda_0, \alpha_0, \beta_0)$). Moreover, (2.13) and (2.14) are satisfied, too. Here we identify the functions $v_j^* \in C^1_{2\pi}$ with functionals

$$x \in C_{2\pi} \longmapsto \int_0^{2\pi} \langle x(t), v_j^*(t)\rangle dt \in \mathbb{R},$$

i.e. with elements of the dual space to $C_{2\pi}$. Finally, (V) implies that

$$\ker \partial_x F(x_0, \lambda_0, \alpha_0, \beta_0) = \operatorname{span}\{S_0 x_0, \dot{x}_0\}.$$

Therefore, (2.4) is satisfied because of

$$\partial_\alpha F(x_0,\lambda_0,\alpha_0,\beta_0) = -S_0 x_0, \;\; \partial_\beta F(x_0,\lambda_0,\alpha_0,\beta_0) = -\dot{x}_0. \tag{5.12}$$

Our last assumption is

(VIII) $\quad \sup\{\operatorname{Re}\xi : \; \xi \in \operatorname{spec}\partial_x F(x_0,\lambda_0,\alpha_0,\beta_0), \; \xi \neq 0\} < 0.$

The assumptions (VI)-(VIII) mean that the Floquet exponent one of the 2π-periodic solution x_0 to equation

$$\beta_0 \dot{x} = f(x, \lambda_0) - \alpha_0 S_0 x \tag{5.13}$$

is semi-simple with multiplicity two and that the absolute values of all other Floquet exponents are smaller than one. This implies that the solutions ξ_0 to (5.1) with $\lambda = \lambda_0$ and $y = 0$ and x_0 to (5.13) are asymptotically $\mathbf{S}^1$–orbitally stable with asymptotic phase (cf. Definition 4.2 and [42]). Hence, the purpose of this chapter is to describe the behavior of an asymptotically $\mathbf{S}^1$-orbitally stable modulated wave solution of an autonomous $\mathbf{S}^1$–equivariant differential equation under a small perturbation of "modulated wave type" $S(\alpha t)y(t)$ with $y(t + \frac{2\pi}{\beta}) = y(t)$ for all t.

This problem occures in the mathematical modeling of the locking behavior of self-pulsating lasers to injected periodically modulated signals, see [4, 20, 33, 38]. In these applications $S(\alpha\tau)y(\beta\tau)$ describes the external injected light (with $\frac{2\pi}{\beta}$-periodic intensity $\|S(\alpha\tau)y(\beta\tau)\| = \|y(\beta\tau)\|$), the modulated wave solution (5.4) is the self-pulsation of the laser (with "optical" frequency α_0 and $\frac{2\pi}{\beta_0}$-periodic intensity $\|S(\alpha_0\tau)x_0(\beta_0\tau)\| = \|x_0(\beta\tau)\|$), and λ describes the internal laser parameters, again.

An application of Theorem 2.1 gives the following result on the persistence of the modulated wave solution (5.4) (more exactly: of the corresponding two parameter family of modulated wave solutions) under perturbations $\lambda \approx \lambda_0$ (but with $y(t) = 0$):

Theorem 5.2 *Suppose* (I) *and* (IV)-(VIII). *Then there exist a neighbourhood* $W \subset \mathbb{R}^n$ *of* λ_0 *and* C^k*-maps* $\hat{u} : W \to C^1_{2\pi}$, $\hat{\alpha} : W \to \mathbb{R}$ *and* $\hat{\beta} : W \to \mathbb{R}$ *with* $\hat{u}(\lambda_0) = 0$, $\hat{\alpha}(\lambda_0) = \alpha_0$, *and* $\hat{\beta}(\lambda_0) = \beta_0$ *such that for all* $\lambda \in W$

$$\int_0^{2\pi} \langle [\hat{u}(\lambda)](t), v_j^* \rangle dt = 0 \quad (j = 1, 2)$$

and that

$$\xi(t) = S(\hat{\alpha}(\lambda)t) \Big[x_0(\hat{\beta}(\lambda)t) + [\hat{u}(\lambda)](\hat{\beta}(\lambda)t) \Big] \tag{5.14}$$

is an asymptotically $\mathbf{S}^1$*-orbitally stable modulated wave solution with asymptotic phase to* (5.1) *with* $y = 0$.

Remark 5.3 It is easy to verify that the map $\lambda \in \mathbb{R}^n \mapsto (\hat{\alpha}(\lambda), \hat{\beta}(\lambda)) \in \mathbb{R}^2$ is a submersion if the map

$$\lambda \in \mathbb{R}^n \longmapsto \left[\int_0^{2\pi} \langle \partial_\lambda f(x_0(t), \lambda_0)\lambda, v_j^*(t) \rangle \, dt \right]_{j=1}^2 \in \mathbb{R}^2$$

is surjective. Hence, in that case no locking between the frequencies $\hat{\alpha}(\lambda)$ and $\hat{\beta}(\lambda)$ occures (that is the generic situation for quasi-periodic solutions of equivariant autonomous differential equations, cf., e.g., [42, 39, 32, 13, 23]).

In order to apply Theorem 2.2 we introduce the set $\mathcal{S}$ of "directions" in the control parameter space $\mathbb{R}^2 \times C^k_{2\pi}$

$$\mathcal{S} := \{(\alpha, \beta, y) \in \mathbb{R} \times \mathbb{R} \times C^k_{2\pi} : |\alpha|^2 + |\beta|^2 + \|y\|_k^2 = 1\}$$

and the locking cones (for $\epsilon_0 > 0$, $(\mu_0^\alpha, \mu_0^\beta, z_0) \in \mathbb{R} \times \mathbb{R} \times C^k_{2\pi}$ and neighbourhoods $W \subset \mathbb{R}^m \times \mathcal{S}$ of $(\lambda_0, \mu_0^\alpha, \mu_0^\beta, z_0)$, cf. (2.12))

$$K(\epsilon_0, \mu_0^\alpha, \mu_0^\beta, z_0, W) :=$$
$$:= \{(\lambda, \hat{\alpha}(\lambda) + \epsilon\mu^\alpha, \hat{\beta}(\lambda) + \epsilon\mu^\beta, \epsilon z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times C^k_{2\pi} : 0 < |\epsilon| < \epsilon_0, (\lambda, \mu^\alpha, \mu^\beta, z) \in W\}.$$

The maps H_j (for $j = 1, 2$, cf. (2.15)), which define the reduced bifurcation equation system, have the following form (cf. (VII) and (5.12))

$$\begin{aligned} H_1(\gamma, \delta, \mu^\alpha, \mu^\beta, z) &= -\mu^\alpha - \textstyle\int_0^{2\pi} \langle S(-\gamma) z(t-\delta), v_1^*(t) \rangle dt, \\ H_2(\gamma, \delta, \mu^\alpha, \mu^\beta, z) &= -\mu^\beta - \textstyle\int_0^{2\pi} \langle S(-\gamma) z(t-\delta), v_2^*(t) \rangle dt \end{aligned} \tag{5.15}$$

for $(\gamma, \delta) \in \mathbb{R}^2$ and $(\mu^\alpha, \mu^\beta, z) \in \mathcal{S}$. Hence, Theorems 5.2 and 6.1 imply the following description of the bifurcation of modulated wave solutions to (5.1) (from certain members of the two parameter family of rotating wave solutions corresponding to (5.4)), their dynamic stability and their asymptotic behavior for y tending to zero:

Theorem 5.4 *Suppose* (I) *and* (IV) – (VIII), *and let* $(\gamma_0, \delta_0) \in \mathbb{R}^2$ *and* $(\mu_0^\alpha, \mu_0^\beta, z_0) \in \mathcal{S}$ *be such that*

$$\begin{aligned} \mu_0^\alpha &= -\int_0^{2\pi} \langle S(-\gamma_0) z_0(t-\delta_0), v_1^*(t)\rangle dt, \\ \mu_0^\beta &= -\int_0^{2\pi} \langle S(-\gamma_0) z_0(t-\delta_0), v_2^*(t)\rangle dt \end{aligned} \tag{5.16}$$

and that the determinant of the martix

$$\begin{bmatrix} \int_0^{2\pi}\langle S_0 S(-\gamma_0) z_0(t-\delta_0), v_1^*(t)\rangle dt & \int_0^{2\pi}\langle S(-\gamma_0)\dot z_0(t-\delta_0), v_1^*(t)\rangle dt \\ \int_0^{2\pi}\langle S_0 S(-\gamma_0) z_0(t-\delta_0), v_2^*(t)\rangle dt & \int_0^{2\pi}\langle S(-\gamma_0)\dot z_0(t-\delta_0), v_2^*(t)\rangle dt \end{bmatrix} \tag{5.17}$$

does not vanish.

Then there exist $\epsilon_0 > 0$, *a neighbourhood* $W \subset \mathbb{R}^n \times \mathcal{S}$ *of* $(\lambda_0, \mu_0^\alpha, \mu_0^\beta, z_0)$, C^{k-1}*-maps* $\hat\gamma : W \to \mathbb{R}$ *and* $\hat\delta : W \to \mathbb{R}$ *with* $\hat\gamma(\lambda_0, \mu_0^\alpha, \mu_0^\beta, z_0) = \gamma_0$ *and* $\hat\delta(\lambda_0, \mu_0^\alpha, \mu_0^\beta, z_0) = \delta_0$ *and a* C^k*-map* $\hat x : K(\epsilon_0, \mu_0^\alpha, \mu_0^\beta, z_0, W) \to C^1_{2\pi}$ *such that the following holds:*

(i) *For* $(\lambda, \alpha, \beta, y) \in K(\epsilon_0, \mu_0^\alpha, \mu_0^\beta, z_0, W)$ *is* $\hat\xi(t, \lambda, \alpha, \beta, y) := S(\alpha t)[\hat x(\lambda, \alpha, \beta, y)](\beta t)$ *a modulated wave solution to* (5.1). *It is asymptotically stable (resp. unstable) if all eigenvalues of* (5.17) *have negative real parts (resp. if one such eigenvalue has a positive real part).*

(ii) *Let* $(\lambda, \mu^\alpha, \mu^\beta, z) \in W$ *be fixed. Then, for* $\epsilon \to 0$,

$$\begin{aligned} &\hat x(\lambda, \hat\alpha(\lambda) + \epsilon\mu^\alpha, \hat\beta(\lambda) + \epsilon\mu^\beta, \epsilon z) \to \\ &\quad \to S(\hat\gamma(\lambda, \mu^\alpha, \mu^\beta, z))[x_0 + \hat u(\lambda)](\cdot + \hat\delta(\lambda, \mu^\alpha, \mu^\beta, z)) \text{ in } C^1_{2\pi}. \end{aligned}$$

Hence, the modulated wave solution $\hat\xi(t, \lambda, \hat\alpha(\lambda) + \epsilon\mu^\alpha, \hat\beta(\lambda) + \epsilon\mu^\beta, \epsilon z)$ *to* (5.1) *tends to the "spatial-temporal" phase shift* $S(\hat\alpha(\lambda)t + \hat\gamma(\lambda, \mu^\alpha, \mu^\beta, z))[x_0 + \hat u(\lambda)](\hat\beta(\lambda)(t + \hat\delta(\lambda, \mu^\alpha, \mu^\beta, z)))$ *of the modulated wave solution* (5.14) *to equation* (5.1) *with* $y = 0$.

Remark 5.5 Theorem 5.4(ii) asserts that, for control parameters $(\lambda, \alpha, \beta, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times C^k_{2\pi}$ of the type $\alpha = \hat\alpha(\lambda) + \epsilon\mu^\alpha$, $\beta = \hat\beta(\lambda) + \epsilon\mu^\beta$ and $y = \epsilon z$ with $\lambda \approx \lambda_0$, $\epsilon \approx 0$, $\mu^\alpha \approx \mu_o^\alpha$, $\mu^\beta \approx \mu_0^\beta$ and $z \approx z_0$, the asymptotically stable modulated wave solution to (5.1) is close to $\xi(t) = S(\alpha_0 t + \gamma_0) x_0(\beta_0(t + \delta_0))$, where the phase shifts γ_0 and δ_0 are determined by the reduced bifurcation equation (5.16). Especially, the phase shifts γ_0 and δ_0 depend on μ_0^α, μ_0^β and z_0, in general (and this dependence is implicitly given by (5.16)). Hence, the frequency locking phenomena cosidered here do not have the so-called phase locking property. On the contrary, it is possible to control the phases of the locked solutions by changing the control parameters (for questions concerning "phase locking" and "phase regulation" see, e.g., [1]).

In the following application of Corollary 2.3 we use the notation (5.5).

Corollary 5.6 *Suppose* (I) *and* (IV) - (VIII), *and let* $\Lambda_* \subseteq \mathbb{R}^n$ *be a subspace such that the map*

$$\lambda_* \in \Lambda_* \mapsto \left[\int_0^{2\pi} \langle \partial_\lambda f(x_0, \lambda_0)\lambda_*, v_j^* \rangle \, dt\right]_{j=1}^2 \in \mathbb{R}^2$$

is surjective.

Then, for each $\alpha \approx \alpha_0$, $\beta \approx \beta_0$ *and* $y \in C_{2\pi}^k$ *with* $\|y\|_k \approx 0$ *such that the matrix*

$$\begin{bmatrix} \int_0^{2\pi} \langle S_0 S(-\gamma) z_0(t-\delta), v_1^*(t) \rangle dt & \int_0^{2\pi} \langle S(-\gamma) \dot{z}_0(t-\delta), v_1^*(t) \rangle dt \\ \int_0^{2\pi} \langle S_0 S(-\gamma_0) z_0(t-\delta), v_2^*(t) \rangle dt & \int_0^{2\pi} \langle S(-\gamma) \dot{z}_0(t-\delta), v_2^*(t) \rangle dt \end{bmatrix}$$

is nonsingular for at least one pair $(\gamma, \delta) \in \mathbb{R}^2$, *there exists a* $\lambda_* \in \Lambda_*$ *near zero such that equation* (5.1) *has a modulated wave solution* $\xi(t) = S(\alpha t) x(\beta t)$ *with* $x \in C_{2\pi}^1$ *and* $x(t) \approx \mathcal{T}$ *for all* t.

Remark 5.7 The assumptions (IV) - (VIII) remain to be valid if one replaces α_0 and x_0 by α_p and x_p (cf. (5.7)) and, hence, $v_1^*(t)$ and $v_2^*(t)$ by

$$v_{1p}^*(t) := S(-\frac{p}{l}t)\left[v_1^*(t) + \frac{p}{l} v_2^*(t)\right] \text{ and } v_{2p}^*(t) := S(-\frac{p}{l}t)(v_1^*(t),$$

respectively. With this new data we get, instead of (5.16), a new reduced bifurcation equation

$$\begin{aligned} \mu_0^\alpha &= -\int_0^{2\pi} \langle S(-\gamma_0 + \tfrac{p}{l}t) z_0(t-\delta_0), v_1^*(t) + \tfrac{p}{l} v_2^*(t) \rangle dt, \\ \mu_0^\beta &= -\int_0^{2\pi} \langle S(-\gamma_0 + \tfrac{p}{l}t) z_0(t-\delta_0), v_2^*(t) \rangle dt. \end{aligned} \tag{5.18}$$

Hence, Theorem 5.4 and Corollary 5.6 describe the frequency locking of the unperturbed modulated wave solution (5.4) with respect to forcings $S(\alpha t) y(\beta t)$ not only for $\alpha \approx \alpha_0$ and $\beta \approx \beta_0$, but also for $\alpha \approx \alpha_p$ and $\beta \approx \beta_0$. Of course, the conditions on the "modulation profile" $y \in C_{2\pi}^k$, in order to get locking with $\alpha \approx \alpha_p$, depend on p, in general. But often for "almost all" modulation profiles the conditions are fulfilled simultaneously for all p. Moreover, in some applications α_0 is much larger than β_0 (in the models for injection locking of self-pulsating lasers $\frac{\alpha_0}{\beta_0}$ is of order 10^5), so that the set $\{\alpha_0 + p\beta_0 : p \in \mathbb{Z}\}$ is in a certain sense "dense" in $\mathbb{R}$. Then one gets frequency locking for all $\beta \approx \beta_0$ and practically for all α and $y \approx 0$.

Remark 5.8 For the problem, considered in Section 4, subharmonic frequency locking does not occure (cf. Remark 4.11). But for the problem considered in this section the situation is quite different.

Let us briefly describe how to get results corresponding to Theorem 5.4 and Corollary 5.6 on forced subharmonic frequency locking of a modulated wave solution (with modulation frequency β_0) to the unperturbed equation under a forcing of "modulated wave type" (with modulation frequency $\beta \approx \frac{p}{q}\beta_0$ with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and p and q relatively prime) into modulated wave solutions (with modulation frequency $\frac{\beta}{q}$). Hence, let us assume (I) and (IV)–(VIII) to be satisfied , and let us look for $2\pi q$-periodic solutions x (with $x(t) \approx \mathcal{T}$ for all t) to equation (5.3) for given control parameters $\lambda \approx \lambda_0$, $\alpha \approx \alpha_0$, $\beta \approx \frac{p}{q}\beta_0$ and $y \in C^k_{2\pi}$ near zero.

We define a $2\pi q$-periodic vector function x_{pq} by $x_{pq}(t) := x_0(\frac{p}{q}t)$. Then, because of assumption (V),

$$\frac{p}{q}\beta_0 \dot{x}_{pq} = f(x_{pq}, \lambda_0) - S_0 x_{pq}.$$

Hence, everything done in this section may be repeated with β_0 and x_0 replaced by $\frac{p}{q}\beta_0$ and x_{pq}, respectively, and by working in spaces of $2\pi q$-periodic vector functions. Especially, all integrals from 0 to 2π have to be replaced by integrals from 0 to $2\pi q$, and, therefore, $v_1^*(t)$ and $v_1^*(t)$ have to be replaced by $\frac{1}{p}v_1^*(\frac{p}{q}t)$ and $\frac{1}{q}v_2^*(\frac{p}{q}t)$, respectively (cf. (VII)). The reduced bifurcation equation (5.16) takes the form

$$\begin{array}{rcl} \mu_0^\alpha & = & -\frac{1}{p}\int_0^{2\pi q}\langle S(-\gamma_0)z_0(t-\delta_0), v_1^*(\frac{p}{q}t)\rangle dt, \\ \mu_0^\beta & = & -\frac{1}{q}\int_0^{2\pi q}\langle S(-\gamma_0)z_0(t-\delta_0), v_2^*(\frac{p}{q}t)\rangle dt, \end{array} \tag{5.19}$$

and one has to look for solutions $\gamma_0 \in [0, 2\pi)$ and $\delta_0 \in [0, 2\pi q)$.

Remark that the results, which one gets in the way described above and which correspond to Theorem 5.4 and Corollary 5.6, are by no means "uniform" with respect to p and q. On the contrary, for (5.19) (with fixed p, q and z_0) to be solvable, the parameter $(\mu_0^\alpha, \mu_0^\beta)$ can vary in a range of diameter of order $(pq)^{-k}$, only. Hence, the corresponding locking cones are "thin" of order $(pq)^{-k}$.

For results concerning the uniformity with respect to p and q of the Liapunov-Schmidt reduction, which is behind Theorem 5.4, see [9].

Remark 5.9 By analogy with Remark 4.12, the generalization of the results of this section to equations of the type

$$\dot{\xi}(\tau) = f(\xi(\tau), \lambda) - S(\alpha\tau)y(\beta\tau, S(-\alpha\tau)\xi(\tau), \lambda) \tag{5.20}$$

is straightforward. In (5.20), the symmetry breaking control parameter y variies in a suitable space of maps from $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$ into $\mathbb{R}^n$ which are 2π-periodic with respect to the first variable.

Acknowledgements

We like to thank U. Bandelow, F. Guyard, R. Lauterbach, B. Sandstede and B. Sartorius for helpful discussions and K. R. Schneider for his continuous interest in our work. Part of this work was presented by the first author during his stay at the Universite Catholique De Louvain in the framework of the HCM network CHRX-CT 94-0555 Nonlinear Boundary Value Problems: Existence, Multiplicity, Bifurcation and Stability of Solutions, and he likes to thank J. Mawhin for his kind hospitality. This research was partially supported by the Deutsche Forschungsgemeinschaft under the grant La 525/4-2 and La 525/4-4.

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