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A sequence of order relations, encoding heteroclinic connections in scalar parabolic PDE

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Abstract

We address the problem of heteroclinic connections in the attractor of dissipative scalar semilinear parabolic equations

$$u_t = u_{xx} + f(x, u, u_x), \qquad 0 < x < 1$$

on a bounded interval with Neumann conditions. Introducing a sequence of order relations, we prove a new and simple criterion for the existence of heteroclinic connections, using only information about nodal properties of solutions to the stationary ODE problem. This result allows also for a complete classification of possible attractors in terms of the permutation of the equilibria, given by their order at the two boundaries of the interval.

1 Introduction

In this paper we investigate the long-time behaviour of scalar semilinear parabolic differential equations

$$u_t = u_{xx} + f(x, u, u_x), \qquad 0 < x < 1, \quad f \in C^2 \tag{1}$$

on a bounded interval with Neumann conditions

$$u_x(0,t) = u_x(1,t) = 0.$$

In the Hilbert space X of x-profiles in $H^2([0,1])$, satisfying the boundary conditions, this equation generates a local C^1-semiflow (see [17]).

$$S_t : \quad u_0 \mapsto u(t) = u(t, \ . \ ) \in X.$$

Note that the profiles contained in our phase space are due to Sobolev embedding in $C^1[0,1]$. Under additional conditions on f, as e.g.

$$f(x, u, 0) \cdot u < 0$$

for large $|u|$, and

$$\partial_x f(x, u, v) + \partial_u f(x, u, v) \cdot v \leq 0$$

for large $|v|$, the semiflow is global and dissipative, i.e. there is a global attractor $\mathcal{A}_f$ which is compact, connected, invariant, and attracts all bounded sets. It consists of all orbits, being defined and uniformly bounded for all positive and negative times

[16], [8]. Due to the gradient structure [30], it can be shown that $\mathcal{A}_f$ contains only the set of equilibria E_f and heteroclinic connections between them [16]:

$$\mathcal{A}_f = E_f \cup \bigcup_{v,w \in E_f} C(v,w)$$

Here, $C(v,w)$ denotes the set of heteroclinic connections from v to w, i.e. orbits $u(t), -\infty < t < \infty$ with $u(t) \to v$ for $t \to -\infty$ and $u(t) \to w$ for $t \to \infty$. If there exists such a connection, we write $v \searrow w$.

A more detailed description of the attractor starts with looking at the stationary problem, i.e. the ODE boundary value problem:

$$u'' + f(x,u,u') = 0, \qquad u'(0) = u'(1) = 0. \tag{2}$$

There has been a lot of investigation about these equilibria solutions, their stability with respect to the semiflow and heteroclinic connections between them (see [16] and references therein). Especially the case of small diffusion has been studied [3]. The case of a cubic nonlinearity has been studied by Chafee and Infante in [9], using bifurcation theory. For this case, a complete description of the equilibria, their Morse-indices, bifurcations and heteroclinic connections can be found in [18].

An important tool to investigate the dynamics on the attractor in a more general situation is the principle of non-increase for the zero-number in the linearized equation. A first version of this result can be found already in the work of Sturm [27]; later it has been extended and refined by Matano [22] and Angenent [1]. It was used to show transversal intersection of stable and unstable manifolds by Henry [18] and Angenent [2]. Brunovsky and Fiedler gave in [6] and [7] for general dissipative $f \equiv f(u)$ and Dirichlet conditions an exact criterion for heteroclinic connections in terms of the zero-numbers of the equilibria. This result shows that the nodal properties of the equilibria are sufficient to determine their PDE connecting orbits.

In [15], Fusco and Rocha pointed out the importance of the permutation ρ_f of the equilibria, given by their order at $x = 0$ and $x = 1$. This permutation contains all information about the nodal properties of the equilibria and allows to treat it in a very systematic and concise way (see Section 3). However, Fusco and Rocha were able to use it for a description of the attractor only for a quite restricted class of such permutations.

In [10], finally, Fiedler and Rocha gave an exact criterion, based on the permutation ρ_f, for connections in the general case $f \equiv f(x,u,u_x)$ which was obtained by Conley-index technique. However, the conditions for a connection derived with this technique are quite involved, and the relation between the permutations and the attractors remains somewhat unclear.

In this paper we introduce a sequence of order relations for the equilibria which has an evident geometrical interpretation, as well for the attractor as for the permutation. In terms of this order relations, we can formulate a simple condition for the existence of heteroclinic connections, similar but even simpler than the condition, given in [7] for the restricted case $f \equiv f(u)$. At the same time, we can show which information about the geometry of the attractor $\mathcal{A}_f$ is necessary to recover the corresponding permutation ρ_f. This allows, using a result in [12], for a complete classification of all possible attractors and the corresponding permutations. The proof is based mainly on bifurcation arguments as used in [15] and a technical result from [28].

The article is organized as follows: In the following section, we introduce our concept of order relations and state the main result. In Section 3, we recall some details about the permutation of the equilibria and how it is related to their nodal properties and invariant manifolds. Moreover, the relation between this permutation and the sequence of order relations will be explained. Section 4 contains the proof of the main result about heteroclinic connections. We conclude with an example and a discussion of possible concepts for the classification of attractors for this type of equation in Section 5.

2 Definitions and statements of main results

Definition 2.1 *For $u(x) \in C^1[0,1]$, we denote by $z(u)$ the number of strict sign changes (zero-number) of $u(x)$ in the interval $[0,1]$.*

Let be H a subset of a phase space X, containing functions from $C^1[0,1]$ which satisfy Neumann boundary conditions. A pair $u_1, u_2 \in H$ with $z(u_1 - u_2) = k$ and all zeroes of $u_1(x) - u_2(x)$ being simple is called k-ordered, and we write

$$u_1 \prec_k u_2,$$

if we have

$$u_1(0) < u_2(0)$$

Note that such an order relation is defined for a dense subset of $X \times X$. The relation $\prec_0$ is the well known partial order, related to the comparison principle. However, for $k > 0$ the relation $\prec_k$ fails to be a partial order in the usual sense. From $u_1 \prec_k u_2$ and $u_2 \prec_k u_3$, we cannot conclude by transitivity that $u_1 \prec_k u_3$ (see Figure 1). Instead u_1 and u_3 may be either not comparable for any k or

$$u_1 \prec_{k'} u_3.$$

for some k' congruent k modulo 2. But since the total order, given by the values at $x = 0$ is still respected, closed loops like

$$u_1 \prec_k u_2 \prec_k u_3 \prec_k u_1$$

are still impossible.

Choosing for H the whole phase space X, the above defined sequence of order relations $(X, \{\prec_k\}_{k \geq 0})$ allows to reformulate the principle of non-increase of the zero-number as a monotonicity principle:

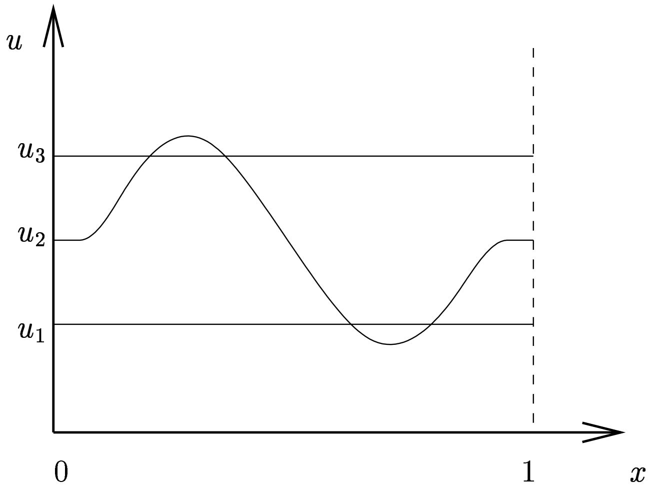

Figure 1: Example with $u_1 \prec_2 u_2 \prec_2 u_3$, but $u_1 \prec_0 u_3$

Proposition 2.2 *The semi-flow S_t of equation (1) respects the sequence of order-relations $(X, \{\prec_k\}_{k\geq 0})$ in the following sense:*

- *If $u_1 \neq u_2$ are in X, then for all positive times except a finite and possibly empty set, $S_t(u_1)$ and $S_t(u_2)$ are k-ordered for some k.*
- *If $u_1 \prec_k u_2$, then we have for almost all $t \geq 0$ either*

$$S_t(u_1) \prec_k S_t(u_2)$$

or $S_t(u_1)$ and $S_t(u_2)$ are k'-ordered for some $k' < k$

Proof: Recall that according to [22] and [2] for the difference $u_1(t) - u_2(t)$ of any two solutions $u_1 \neq u_2$ to (1) the following holds true: $z(u_1(t) - u_2(t))$ is finite for any positive t, non increasing in t, and drops strictly at a discrete set of values of t, where the difference of the two profiles evolves a multiple zero

$$u_1(x,t) = u_2(x,t) = \partial_x u_1(x,t) = \partial_x u_2(x,t) = 0$$

for some $x \in [0,1]$.

Assume at a time t_0 the two trajectories $S_t(u_1)$ and $S_t(u_2)$ stop to be k-ordered. Then either the zero-number changes, but it can only drop, or the order at $x = 0$ changes. But due to Neumann boundary conditions, this also leads to a double zero at $x = 0$, and hence to a dropping of k. □

Definition 2.3 *Let be the set H as in Definition 2.1 and finite, and the pair $u_1, u_2 \in H$ k-ordered with $u_1 \prec_k u_2$. We call the pair u_1, u_2 k-adjacent, and write*

$$u_1 \prec\!\!\prec_k u_2,$$

if there is no third element $u_3 \in H$ *with*

$$u_1 \prec_k u_3 \prec_k u_2.$$

A sequence $\{ v_{j_1}, v_{j_2}, \dots, v_{j_r}\} \subset H$, $r \geq 2$,*is called a k-order-chain, if*

$$v_{j_1} \prec\!\!\prec_k v_{j_2} \prec\!\!\prec_k \dots$$

If it is not a proper subset of any other k-order-chain, we call it a maximal k-order-chain.

Note that each k-order-chain, finally, carries a total order in the usual sense; however, two not-adjacent members of a k-order-chain need not to be k-ordered. For each $k \geq 0$ the union of all k-order-chains carries a partial order, induced by the total order on each order-chain which are consistent as we mentioned before.

These definitions can be related to the problem of heteroclinic connections in (1) as follows: For H, we choose the set E_f of all equilibria solutions to (1). Due to Sobolev embedding, all the x-profiles are in $C^1[0,1]$. If we assume in addition hyperbolicity of all equilibria, then E_f is finite. Moreover, any pair of equilibria $e_1, e_2 \in E_f$ is k-ordered for some k. Now, the following theorem can be formulated:

Theorem 2.4 *Two hyperbolic equilibria solutions* $v, w \in E_f$ *with* $z(v-w) = k$ *have a heteroclinic connection if and only if they are k-adjacent.*

This condition can be checked easily from a plot of the equilibria. Checking for a heteroclinic connection between two equilibria $v, w \in E_f$, one even needs only to look at those equilibria which are in between v and w at both $x = 0$ and $x = 1$. To decide, however, which of both is the source and which is the target, one needs additional information: In Lemma 4.5 we will show that any maximal order-chain consists of alternating sources and targets, beginning and ending both with a target equilibrium. An other possibility to resolve this question is to use a result of Fiedler and Rocha which will be discussed in Section 3. It is possible to compute the Morse-indices from the nodal properties of all equilibria (i.e. the permutation of the equilibria). The Morse-Smale property yields that the equilibrium with higher index has to be the source.

3 Nodal properties, meandric permutations and invariant manifolds

In this section we recall briefly how the nodal properties of the equilibria are encoded by a permutation, given by the order of the equilibria at both ends of the interval [0, 1]. Moreover, we explain the relation between this permutation and the sequence of order relations $(E_f, \{\prec_k\}_{k\geq 0})$, given in the previous section. Finally, we recall

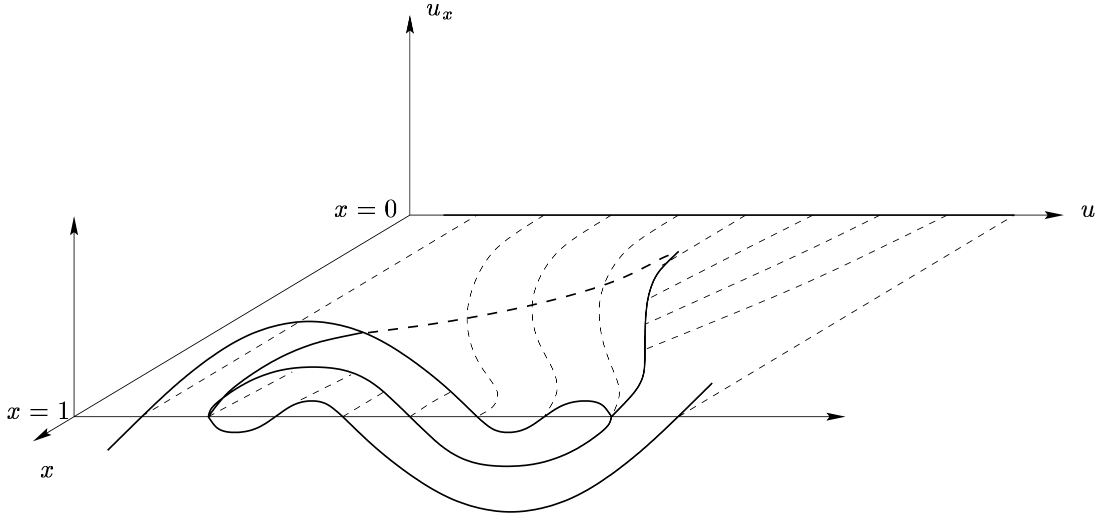

Figure 2: surface $S(x, \alpha)$

some fundamental results about the structure of the attractors. They show, how the nodal properties of Sturm-Liouville eigenfunctions, together with the principle about the zero-number dropping, can be used to determine the nodal properties in the invariant manifolds, building up the attractor.

With $E_f = \{v_1, \ldots, v_n\}$ we denote the set of all equilibria for (1), i.e. all solutions to equation (2), which will be assumed to be all hyperbolic. Due to dissipativity, this will be a finite set. Taking their order at $x = 0$

$$v_1(0) < v_2(0) < \ldots < v_{N-1}(0) < v_N(0),$$

we can define the permutation ρ according to the values at $x = 1$:

$$v_{\rho(1)}(1) < v_{\rho(2)}(1) < \ldots < v_{\rho(N-1)}(1) < v_{\rho(N)}(1).$$

An important feature of this permutation can be seen as follows: Consider all trajectories $u(x)$, $x \in [0, 1]$ of the spatial dynamics (2) which satisfy the first boundary condition $u_x(0) = 0$. This is a one-parametric family of curves, parametrized by $u(0) = \alpha$ and forms a smooth surface $S(x, \alpha)$ in the extended phase space (u, u_x, x) (see Figure 3). The intersection points of the curve $S(1, \alpha) := \gamma(\alpha)$ with the straight line $\{x = 1, u_x = 0\}$ lie on trajectories satisfying also the boundary condition at $x = 1$. The order of these solutions along the curve $\gamma(\alpha)$ is the same as along the line $\{x = 0,\ u_x = 0\}$. Thus, the permutation ρ_f of the equilibria is determined only by $\gamma(\alpha)$. This curve has no self-intersections and hence the permutation ρ_f is a so called planar or meandric permutation. Such permutations were first described by V.I. Arnol'd in [4]. They give rise to a lot of interesting questions and have been studied also from a pure combinatorial point of view (see [20],[21],[26]).

One can easily prove that the condition for the equilibria to be hyperbolic makes the corresponding intersection point of the curve $\gamma(\alpha)$ transversal. Moreover, the dissipativity condition on f leads to $\gamma(\alpha) \to \pm\infty$ for $\alpha \to \pm\infty$. Hence, there has to be an odd number of hyperbolic equilibria. Obviously the permutation determines the curve up to a diffeomorphism of the phase plane.

The importance of these permutations for a description of the attractors of equation (1) has been discovered by G. Fusco and C. Rocha in [15]. Later they have been used in several papers on this subject ([10], [11], [12], [13], [24], [28]). In [10] it has been shown, how from these meandric curves we can read off two important combinatorial invariants of the set E_f:

- Counting the number of clockwise half-turns, performed by a tangent vector to the curve $\gamma(\alpha)$ along a path from outside the region of intersections to an intersection point v_j, we obtain the winding numbers $i(v_j)$, $j = 1, \dots, n$. The winding number $i(v_j)$ has been shown to be equal to the Morse-index (i.e. dimension of the unstable manifold) of the equilibrium v_j.

- The number of clockwise half-turns, performed by a line, connecting an intersection point v_j with a point, moving along the meandric curve from outside the region of intersections to an intersection point v_k, is equal to the number of zeros $z(v_j - v_k)$ of the difference of the corresponding x-profiles of the equilibria.

Moreover it has been shown in [12] that indeed all meandric permutations with non-negative winding numbers can be realized as the configuration of all the solutions to (2) by an appropriate choice of $f(x, u, u_x)$.

We will show now that there is a simple relation between the permutation ρ_f of the equilibria and the sequence of order-relations $(E_f, \{\prec_k\}_{k\geq 0})$.

Lemma 3.1 *Let be E_f the set of all equilibria, and ρ_f the corresponding permutation. Then the whole sequence of order-relations $(E_f, \{\prec_k\}_{k\geq 0})$ can be obtained from ρ_f.*

Proof: Recall that for any pair of equilibria $v, w \in E_f$, the zero number $z(v - w)$ can be obtained from the meandric permutation ρ_f as the number of positive clockwise half-turns around v, performed by the meandric curve along the curve segment between the first equilibrium v_1 and w. Together with the order of the equilibria at $x = 0$ which is obviously given by the permutation this allows to calculate the whole sequence of order relations $(E_f, \{\prec_k\}_{k\geq 0})$. □

Note that there are of course lots of abstract sequences of order relations which cannot be realized by a set of functions $H \subset C^1[0, 1]$. Moreover, for an arbitrary finite set of functions $H \subset C^1[0, 1]$, its sequence of order relations $(H, \{\prec_k\}_{k\geq 0})$ in general cannot be obtained from some meandric permutation ρ as in the previos Lemma. However, the following Lemma shows that, if a sequence of order relations originates from a set of all equilibria E_f, i.e. there exists a realizing meandric permutation ρ_f, then this permutation can of course be easily recovered from these order relations. Moreover, we show that such realizable sequences of order relations are already determined by the adjacency relations. This is of course not true for general abstract sequences of order relations.

Lemma 3.2 *(1) Let be E_f the set of all equilibria and $(E_f, \{\prec_k\}_{k\geq 0})$ the corresponding sequence of order relations. Then the permutation ρ_f can be calculated from these order relations.*

(2) If for two different nonlinearities f_1, f_2 there is a bijection $\sigma : E_1 \longmapsto E_2$, such that

$$v \prec\!\!\prec_k w \Longleftrightarrow \sigma(v) \prec\!\!\prec_k \sigma(w),$$

then the corresponding meandric permutations ρ_1, ρ_2 are equal.

Proof: Any pair of equilibria $v, w \in E_f$ is k-ordered for some k, and from this order we can recover their order at $x = 0$ and, taking into account whether k is even or odd, also the order at $x = 1$ (for even k the order has to be the same as at $x = 0$, for odd k the inverse). This is clearly sufficient for recovering the permutation and proves part (1).

In order to prove part (2), we make the following assertion: Any sequence of order-relations $(E, \{\prec_k\}_{k\geq 0})$, originating from a meandric permutation has the property that for any pair of equilibria $v, w \in E$ there is at least one order-chain, containing both v and w. Using this assertion, the Lemma follows immediately: Again, for any pair of equilibria from their order in the order-chain, we can recover their order at $x = 0$ and $x = 0$. Due to the bijection, these orders have to be the same for $\sigma(v)$ and $\sigma(w)$. Hence $\rho_1 = \rho_2$.

Now, we prove the assertion. Suppose the pair of equilibria v, w has zero-number $z(v - w) = k$ and $v \prec_k w$. If they are in addition k-adjacent, they obviously form a k-order-chain. Otherwise, by definition, there is a third equilibrium $\tilde{v}$ with

$$v \prec_k \tilde{v} \prec_k w.$$

By induction, we can conclude that there are k-order-chains from v to $\tilde{v}$, as well as from $\tilde{v}$ to w. Together, they form a k-order-chain from v to w. □

The last Lemma allows also conclusions about the recovering of the permutation from some information about the heteroclinic connections in a given attractor. Due to Theorem 2.4 the adjacencies of equilibria correspond exactly to the heteroclinic connections in the attractor. Hence Lemma 3.2 can be interpreted as follows: If for a given attractor we know for all pairs of connected equilibria $v \searrow w$ their order at $x = 0$, and the zero-number $z(v - w)$, then we can uniquely determine the corresponding permutation ρ_f. The question how this result can be used for a classification of attractors and corresponding permutations will be discussed in detail in Section 5.

In the following propositions we recall some fundamental results which will be used later. They describe the invariant manifolds, their transversal intersections, and the nodal properties of the solutions contained therein:

Proposition 3.3 *Let v be a hyperbolic equilibrium with Morse-index $i(v) = n$. Then we have the (strong-)unstable manifolds*

$$W_1^u(v) \subset W_2^u(v) \subset \ldots \subset W_n^u(v) = W^u(v), \tag{3}$$

where each $W_j^u(v)$ has the dimension j. The span $\langle \phi_0, \ldots, \phi_{j-1} \rangle$ of the first j eigenfunctions is in v tangent to $W_j^u(v)$ and parametrizing it globally. An eigenfunction ϕ_k has exactly k zeros. For $u_1 \neq u_2$ in the closure $\overline{W_j^u(v)}$, we have

$$z(u_1 - u_2) < j$$

Analogously we have the infinite dimensional (strong-)stable manifolds

$$\ldots \subset W_{n+2}^s(v) \subset W_{n+1}^s(v) \subset W_n^s(v) = W^s(v). \tag{4}$$

Here, each $W_k^s(v)$ has codimension k. The tangent space at v is the span $\langle \phi_k, \phi_{k+1}, \ldots \rangle$ of all but the first k eigenfunctions. For $u_1 \neq u_2 \in \overline{W_k^s(v)}$, we have

$$z(u_1 - u_2) \geq k.$$

All intersections of (strong-)stable and (strong-)unstable manifolds are transversal. Hence,

$$W_j^u(v) \pitchfork W_k^s(w) =: C_{j,k}(v, w)$$

is a embedded submanifold and, if it is not empty, of dimension $j - k$.

The existence of the manifolds follows from standard theorems [17] and classical Sturm-Liouville theory. The condition on the zero-numbers was obtained in [5], using [22] and [18]. For results on global parametrization see [19] and [24]. Transversality was proved in [18] and for strong stable and unstable manifolds in [15].

We will use also the following result from [28], showing how for a connection $v \searrow w$ the zero number $z(v - w)$ determines in which strong-unstable manifolds of v and strong-stable manifolds of w the heteroclinic orbits are contained:

Proposition 3.4 *Let v, w be two equilibrium solutions of (1) with a heteroclinic connection $v \searrow w$. Then the $(j - k)$-dimensional manifold $C_{j,k}(v, w)$ is nonempty, if and only if*

$$\begin{aligned} z(v - w) < \quad & j \quad \leq i(v) \\ i(w) \leq \quad & k \quad \leq z(v - w). \end{aligned}$$

4 Proof of the main theorem

To prove Theorem 2.4, we proceed as follows: In Proposition 4.1 we recall that the necessity of adjacency for heteroclinic connections is an immediate consequence of the zero-number dropping principle (see also [7]). Then, we show a technical lemma about combinatorial properties of meandric curves, leading to a distinction of several cases. Finally, we proof the theorem, mainly by investigating pitchfork and saddle-node bifurcations in the attractor.

Proposition 4.1 *If two hyperbolic equilibria $v, w \in E_f$ with $v \prec_k w$ have a heteroclinic connection, then they are adjacent.*

Proof: Assume there is a heteroclinic orbit $u(t)$, $-\infty < t < \infty$, connecting from v to w, and $v \prec_k w$ are not adjacent. Then by definition there is a $\tilde{v}$ with

$$v \prec_k \tilde{v} \prec_k w.$$

For large negative t, $u(t)$ is close to v and we have $u(t) \prec_k \tilde{v}$. By Proposition 2.2 we have for $T > t$ either $u(T) \prec_k \tilde{v}$ or $u(T)$ and $\tilde{v}$ are k'-ordered for some $k' < k$. This clearly contradicts to $\tilde{v} \prec_k u(T)$ which is true for large T, when $u(T)$ becomes close to w. □

Definition 4.2 *A pair of intersection points (nodes) in a meandric curve is called a* short arc*, if the nodes are subsequent both along the curve and the straight line.*

Lemma 4.3 *Let be ρ a meandric curve, $v, w \in E_\rho$, and $v \neq w$. Then one of the following assertions is true*

(1) There is a short arc v_j, v_{j+1} with

$$\{v, w\} \cap \{v_j, v_{j+1}\} = \emptyset$$

(2) At least one of the two nodes, say v, is contained simultaneously in two short arcs.

(3) ρ is a spiral and v, w are the predecessor and successor of the central node v_c(see Figure 3).

Proof: First, note that any meander ρ has at least one short arc in the upper and one in the lower half-plane. If these are the only short arcs, then ρ is obviously a spiral. If v or w is the center of the spiral, we are in case (2); otherwise if v or w is not contained in one of the two short arcs, one of these has to be disjoint from $\{v, w\}$ and we are in case (1). The only remaining possibility for the spiral is now as described in (3).

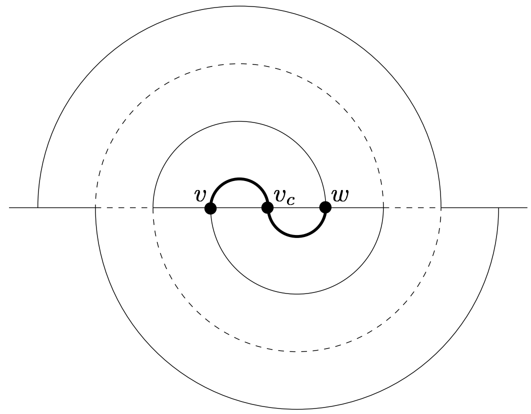

Figure 3: The spiral has only two short arcs

Now, we consider the case with three or more short arcs. If each of v and w meets only one short arc, then at least one of those is disjoint from $\{v, w\}$ and we are in case (1). Finally, if one of the nodes is contained simultaneously in two short arcs, we are in case (2). □

Proof of Theorem 2.4: Let be ρ a meandric permutation of $2N+1$ equilibria which is a minimal counterexample for the theorem. This means there are equilibria $v, w \in E_\rho$ which are adjacent but not connected (That a connection implies adjacency has been shown in Proposition 4.1). At the same time, for all meandric permutations $\hat{\rho}$ of $2N-1$ equilibria the theorem is assumed to hold true. If $N = 1$, this is trivially satisfied.

We now want to perform the proof by induction, following the distinction of cases given in Lemma 4.3. Indeed, for case (1) we can reduce the number of equilibria by two, removing the short arc $\{v_j, v_{j+1}\}$ by a saddle-node bifurcation (see Figure 4). The existence of a corresponding family of nonlinearities $f_\mu(u, u_x, x)$ is an immediate

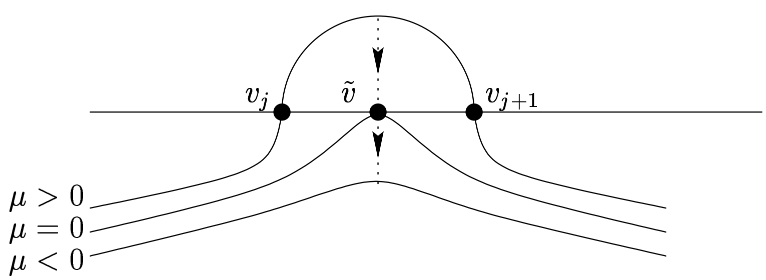

Figure 4: Removing a short arc by a saddle-node bifurcation

consequence of the realization result in [12]. For case (2), the induction step can be performed by a pitchfork bifurcation. This scenario has already been studied by Fusco and Rocha (see [15]) and we can refer to their results. In case (3), finally, the contradiction is obvious, since

$$z(v - v_c) = z(w - v_c) = z(v - w) =: h,$$

which implies that v and w are not adjacent, because we have

$$v \prec_h w \text{ and } v \prec\!\!\prec_h v_c \prec\!\!\prec_h w.$$

Before we study these bifurcations in detail, we point out that a local bifurcation can influence an existing connection $v \searrow w$ only in cases, where at the bifurcation v, w, or some intermediate equilibrium $\tilde{w}$ with $v \searrow \tilde{w} \searrow w$ is non-hyperbolic. This argument has been shown by Henry in ([18], Proof of Thm. 9) and was used also by Fusco and Rocha in [15].

Now, let us assume that the permutation ρ contains two subsequent short arcs v_{j-1}, v_j and v_j, v_{j+1}. Hence it can be obtained from the permutation $\hat{\rho}$ with the $2N-1$ nodes $E_{\hat{\rho}} = \{v_1, \dots, v_{j-1}, v_{j+2}, \dots, v_{2N+1}\}$ by a pitchfork bifurcation at $v_{j-1} \in E_{\hat{\rho}}$, replacing the single node v_{j-1} by three nodes, connected with two subsequent short arcs. We assume moreover that for the bifurcation parameter $\mu = 0$ the eigenvalue λ_k, corresponding to an eigenfunction with k zeroes, becomes critical and for $\mu > 0$ the two new equilibria are generated. The result in [15] shows that all the connections which are for $\mu < 0$ contained in the non-critical directions of v_{j-1}

$$H(v_{j-1}) := W^s_{k+1}(v_{j-1}) \cup W^u_k(v_{j-1})$$

persist for $\mu > 0$ in the corresponding manifolds of each of v_{j-1}, v_j and v_{j+1}. Recall that the subscripts at the manifolds denote codimension and dimension, respectively. The zero-numbers in the manifolds are given in Proposition 3.3.

Note that for μ sufficiently close to zero all zero numbers to the remaining equilibria persist:

$$z_{\hat{\rho}}(v_{j-1} - v_r) = z_\rho(v_b - v_r) \tag{5}$$

where $b \in \{j-1, j, j+1\}$ and $r \in \{1, \dots, j-2, j+2, \dots, 2N+1\}$. In addition, we have

$$z(v_{j-1} - v_j) = z(v_j - v_{j+1}) = z(v_{j-1} - v_{j+1}). \tag{6}$$

To cover case (2) of Lemma 4.3, we take $v = v_j$ and check whether there can exist some $w \in E_\rho$ such that v_j and w are adjacent for $\mu > 0$, but not connected. If $z_\rho(v_j - w) \neq k$ then the adjacency of v_j and w for ρ implies adjacency of v_{j-1} and w for $\hat{\rho}$. By induction this implies a connection of v_{j-1} with w for $\mu < 0$. Due to the zero-number and Proposition 3.4, such a connection is contained in $H(v_{j-1})$ and hence for $\mu > 0$ it will be inherited by v_j, as explained above.

In the case $z_\rho(v_j - w) = k$, we have from (5) and (6) immediately either non-adjacency

$$w \prec_k v_{j-1} \prec_k v_j \quad \text{or} \quad v_j \prec_k v_{j+1} \prec_k w,$$

or $w \in \{v_{j-1}, v_{j+1}\}$. In the last case, the existence of a connection follows from elementary bifurcation theory (for details, see again [15]).

To finish the proof, we have to treat case (1) of Lemma 4.3. So, assume again by changing the parameter μ we pass from the permutation $\hat{\rho}$ with the $2N-1$ nodes $E_{\hat{\rho}} = \{v_1, \ldots, v_{j-1}, v_{j+2}, \ldots, v_{2N+1}\}$ for $(\mu < 0)$ to the permutation ρ with $E_\rho = \{v_1, \ldots, v_{2N+1}\}$, inserting the two equilibria v_j, v_{j+1} now by a saddle-node bifurcation. At $\mu = 0$, we have a single non-hyperbolic equilibrium $\tilde{v}$. Again λ_k is the critical eigenvalue of $\tilde{v}$, corresponding to an eigenfunction with k zeroes. Recall that $\{v, w\}$ is assumed to be disjoint from $\{v_j, v_{j+1}\}$. Obviously, zero-numbers of pairs of equilibria in $E_{\hat{\rho}}$ do not change during the bifurcation. From this we conclude that adjacency of v and w for ρ implies their adjacency also for $\hat{\rho}$ and hence by induction a connection, say $v \searrow w$, exists for $\mu < 0$. As we pointed out above, this connection either persists during the bifurcation and we are finished, or we have at $\mu = 0$

$$v \searrow \tilde{v} \searrow w,$$

since $\tilde{v}$ is the only non-hyperbolic equilibrium. With the same transversality arguments as in the pitchfork case (see [15]), it can be shown that connections in the non-critical manifolds

$$H(\tilde{v}) := W^s_{k+1}(\tilde{v}) \cup W^u_k(\tilde{v})$$

are inherited by the corresponding manifolds of both v_j and v_{j+1}. For a generic saddle-node, the one dimensional local centre-manifold $W^c_{loc}(\tilde{v})$ (cf. [17],[18]) consists of two branches, one stable and one unstable. Since the manifold can be parametrized by the corresponding eigenvector, it is obvious that one branch contains functions $u \in X$ with $u \prec_k \tilde{v}$, whereas in the other we have $\tilde{v} \prec_k u$. The connections in each branch are inherited only by one of the hyperbolic equilibria v_j and v_{j+1}. The transversal intersection of stable and unstable manifolds at connections with non-hyperbolic equilibria has been shown in [18], Theorem 8.

Note that again for sufficiently small μ, we have

$$z(\tilde{v} - v_r) = z(v_j - v_r) = z(v_{j+1} - v_r)$$

for all $r \in \{1, \ldots, j-1, j+2, \ldots, 2N+1\}$, and of course $z(v_j - v_{j+1}) = k$.

Now, we have to distinguish several cases: If $z(v - \tilde{v}) = z(w - \tilde{v}) = k$, then adjacency of v and w breaks down in the following way: Let be $u_v(t)$ an orbit connecting from v to $\tilde{v}$ and $u_w(t)$ from $\tilde{v}$ to w. For large t, $u_v(t)$ and $u_w(-t)$ are contained in different branches of $W^c_{loc}(\tilde{v})$, i.e. say

$$u_v(t) \prec_k \tilde{v} \prec_k u_w(-t)$$

(or the reversed order, of course). Applying Proposition 2.2 yields

$$v \prec_k \tilde{v} \prec_k w.$$

From

$$z(v - w) \leq z(v - \tilde{v}) = k = z(\tilde{v} - w) \leq z(v - w),$$

we obtain $z(v - w) = k$ and hence $\tilde{v}$ destroys k-adjacency for $\mu = 0$. This is preserved for $\mu > 0$, replacing $\tilde{v}$ by v_j or v_{j+1}.

The second case is $z(v-\tilde{v}) \neq k$. This implies also

$$z(v-v_j) \neq k \neq z(v-v_{j+1}). \tag{7}$$

Any connection which starts or ends at $\tilde{v}$ is inherited for $\mu > 0$ at least by one of the equilibria v_j and v_{j+1}. From the zero-numbers (7) and Proposition 3.4 we conclude that the connection $v \searrow \tilde{v}$ is contained in the non-critical manifolds and hence persists for both v_j and v_{j+1}. This allows to establish a connection $v \searrow w$ by a cascade $v \searrow v_\nu \searrow w$, even though for $z(w-\tilde{v}) = k$, w may be connected with only one equilibrium $v_\nu \in \{v_j, v_{j+1}\}$. The case $z(w-\tilde{v}) \neq k$ and $z(v-\tilde{v}) = k$ can be treated analogically. □

Remark 4.4 *This theorem covers and substantially simplifies the results of Fiedler and Rocha in [10], obtained by Conley-index techniques. However the only tool which was used above to establish the existence of heteroclinic connections is their generation at pitchfork bifurcations (cf. [15]) and a transitivity argument for connections in Morse-Smale flows (see e.g. [18], p. 191). Also the proof of Proposition 3.4 (cf. [28]), which we used here, does not rely essentially on the results in [10].*

We want to demonstrate now the application of this theorem with an example: Consider the configuration of equilibria, given in Figure 5, together with the cor-

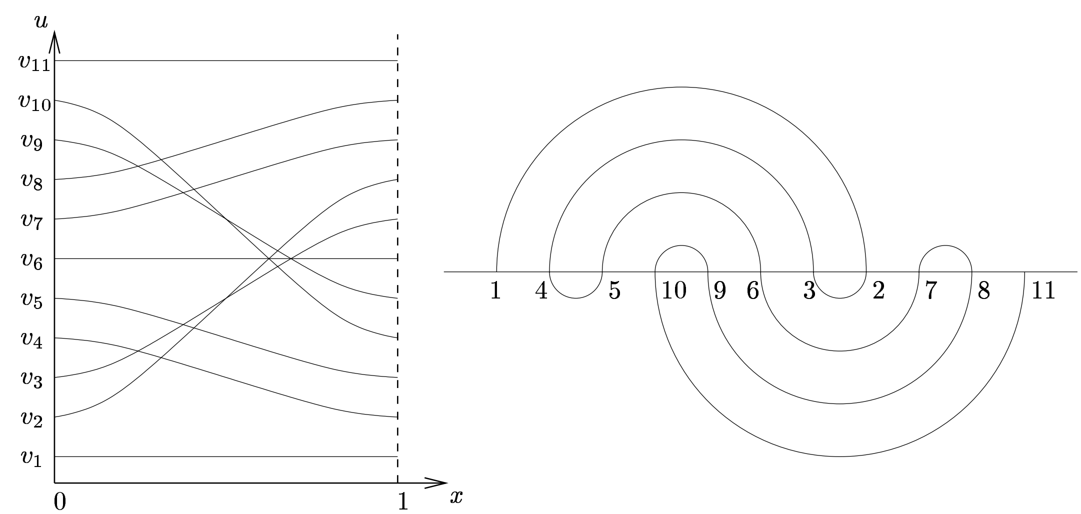

Figure 5: Permutation $\rho_f = (2\,4\,10\,8)(3\,5\,9\,7)$

responding meandric permutation $\rho_f = (2\,4\,10\,8)(3\,5\,9\,7)$. It is now easy to figure out the adjacencies. We have drawn in Figure 6 for each k which appears as the zero-number for some pair of equilibria the union of all k-order-chains. Each arrow indicates one adjacency relation. From this, one gets immediately a picture of the attractor (Figure 7). Note that we could have been started as well with a

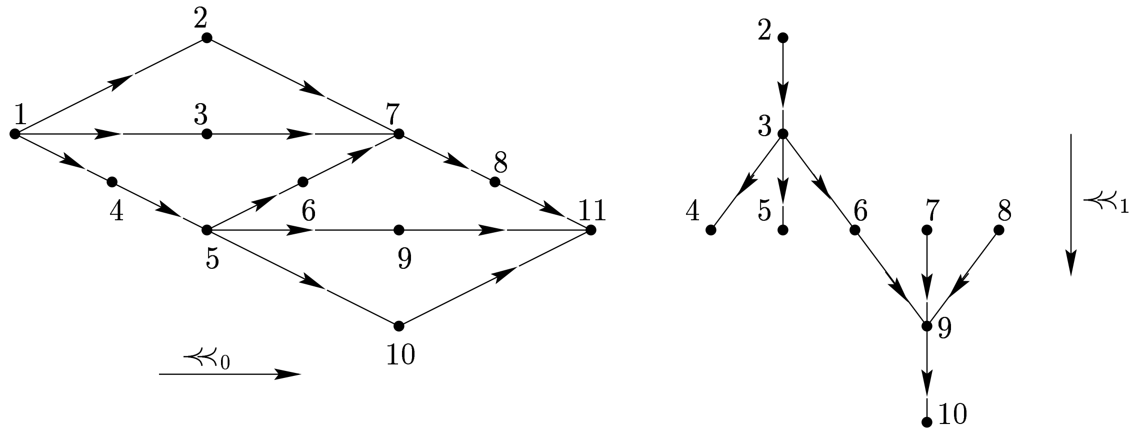

Figure 6: Order-Chains for the order relations $\lll_0$ and $\lll_1$

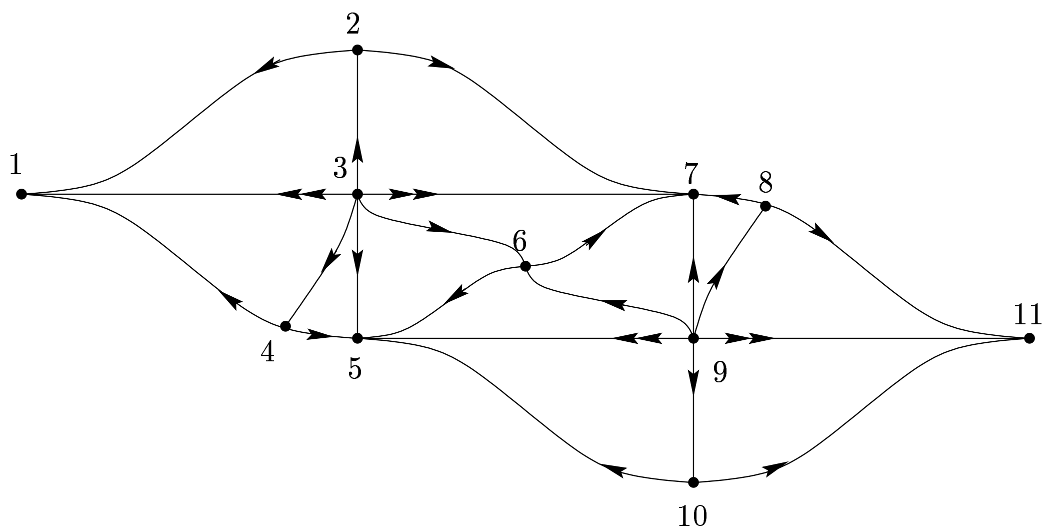

Figure 7: Flow on the corresponding attractor

sufficiently detailed sketch of the attractor, then extract the order-chains, and finally check whether the derived permutation is meandric and hence realizing the suggested attractor.

In Figure 7 we make use of a further result from [28]: If there is a heteroclinic connection $v \searrow w$ and $z(v - w) = h$, then the intersection

$$C_{h+1,h}(v,w) = W^u_{h+1}(v) \;\overline{\pitchfork}\; W^s_h(w)$$

contains exactly one heteroclinic orbit. Due to Proposition 2.2 and 3.3, exactly on these orbits in the attractor the semi-group acts monotonically with respect to the corresponding order relation $\prec_h$. We have drawn here for each set of connecting orbits $C(v,w)$ only this single heteroclinic orbit. This leads according to Theorem 2.4 to a one to one correspondence between the arrows in Figure 7 and the arrows in Figure 6, denoting adjacency. Note that the arrows in Figure 7 have to point in contrast to the arrows of Figure 6 in alternating directions. This observation is explained by the following Lemma:

Lemma 4.5 *Let ρ be a meandric permutation with non-negative winding numbers and $(E_\rho, \{\prec_k\}_{k\geq 0})$ the induced sequence of order-relations. Then for a maximal k-order-chain $S = \{s_1, \ldots, s_r\} \subseteq E_\rho$ we have:*

(1) The length r of the maximal order-chain S is odd

(2) For every $j \in \{1, \ldots, r\}$ we have $i(s_j) > k$, if j is even, and $i(s_j) \leq k$, if j odd.

(3) $s_1 \swarrow s_2 \searrow s_3 \swarrow \ldots \searrow s_r$

Proof: Everything follows immediately from the assertion that $i(s_0), i(s_r) \leq k$: Indeed, starting from s_0, we get step by step from Theorem 2.4 and Proposition 3.4 the conditions (2) on the indices and the connections $s_1 \swarrow s_2 \searrow s_3 \swarrow \ldots$. Ending with $i(s_r) \leq k$, forces the length r to be odd.

The assertion $i(s_0) \leq k$ can be verified as follows: For any v_j with $i(v_j) > k$, we consider the function

$$z_{v_j}(v_\nu) := z(v_j - v_\nu)$$

with ν ranging from 1 to j. Obviously, we have $z_{v_j}(v_{j-1}) \geq i(v_j) - 1$, and $z_{v_j}(v_1) = 0$ (cf. Section 3). Since the function z_{v_j} changes its values for subsequent equilibria by at most ± 1, it takes all values from zero to $i(v_j) - 1$. Hence we have also

$$z_{v_j}(v_{\tilde{j}}) = k$$

for some $\tilde{j} < j$, i.e $v_{\tilde{j}} \prec_k v_j$. Consequently, v_j cannot be the first element s_0 in a maximal k-chain. Obviously, $i(s_r) \leq k$ follows in the same way. □

Note that, looking only on the attractor, it is not evident which pairs of equilibria can be brought together in a saddle-node bifurcation. The following Lemma shows how this is determined locally by the structure of the order-chains:

Lemma 4.6 *Let $(E_f, \{\prec_k\}_{k\geq 0})$ the set of all equilibria, together with its sequence of order relations. Then the pair $\{v, w\} \subset E_\rho$, $v \prec_k w$, is a short arc in ρ_f, if and only if*

$$\begin{array}{rcl} Adj_k^{+}(v) & = & \{\tilde{v} \in E_\rho |\, v \prec\!\!\prec_k \tilde{v}\} = \{w\} \\ Adj_k^{-}(w) & = & \{\tilde{w} \in E_\rho |\, \tilde{w} \prec\!\!\prec_k w\} = \{v\}. \end{array} \qquad (8)$$

Proof: If $\{v, w\}$ is assumed to be a short arc with $v \prec_k w$, then (8) follows immediately from the saddle-node bifurcation scenario (recall from the proof of Theorem 2.4).

Now, we assume (8) to be satisfied for $\{v, w\}$. Since (8) implies in particular $v \prec\!\!\prec_k w$, we get from Lemma 4.5 information about the indices $i(v)$ and $i(w)$. One of both, say $i(v)$, is greater than k, the other one smaller or equal. Since $\mathrm{Adj}_k{}^+(v)$ is assumed to contain only one element, we can conclude that $i(w) = k$: Indeed, from [28], Lemma 4.6, 4.8, it follows that for any $u \in E_f$ and $l < i(u)$, $\nu \in \{+,-\}$, the set $\mathrm{Adj}_l{}^\nu(u)$ contains at least one equilibrium u_l^ν with $i(u_l^\nu) = l$. Applying this, we obtain that $v_k^+ = w$, and hence $i(w) = k$.

Moreover, we can conclude that

$$z_v(\tilde{v}) := z(v - \tilde{v}) > k \tag{9}$$

for all $\tilde{v} \in E_f$ with $v(0) < \tilde{v}(0) < w(0)$: Note that the first $\tilde{v}$, violating this condition has $z_v(\tilde{v}) = k$ and hence $v \prec\!\!\prec_k \tilde{v}$, but $\mathrm{Adj}_k{}^+(v) = \{w\}$. With $\tilde{w}$ we denote now the predecessor of w along the meandric curve. From (9) we obtain that $z_v(\tilde{w}) > k$ and since $z_v(\tilde{w})$ may differ from $z_v(w) = k$ at most by ± 1, we get $z_v(\tilde{w}) = k + 1$. The change of the function z_v along the subsequent nodes $\tilde{w}$ and w implies that v has to be between them at $x = 1$. Hence we get

$$\tilde{w}(1) < v(1) < w(1), \tag{10}$$

if we assume that k is even; for odd k the inequality (10) is valid in reversed order. In any case, since $z(w - \tilde{w})$ may differ from $i(w) = k$ at most by ± 1 and, due to (10) should be congruent $z(v - w)$ modulo 2, this yields

$$z(w - \tilde{w}) = k.$$

Using (8) and the same arguments as above, we obtain $\tilde{w} = v$. Thus, the equilibria v and w are subsequent along the meandric curve.

But there is a well-known duality between the meandric curve and the straight line: Stretching the curve by a homotopy of the plane to a straight line and simultaneously deforming the straight line into a curve, gives the inverse permutation (see [10]), preserving adjacency. In Section 5, we will discuss this transformation more detailed. Applying this transformation to the arguments above, we obtain that v and w are also subsequent along the straight line. This shows that $\{v, w\}$ is indeed a short arc. □

5 Classification of the attractors

Lemma 3.1 and 3.2 allow for a complete classification of all possible attractors in the following sense:

Definition 5.1 *Two attractors $\mathcal{A}_f, \mathcal{A}_g$ are called order-equivalent, if there exists a bijection $\sigma : E_f \longrightarrow E_g$ of the equilibria such that for all $k \geq 0$, $v, w \in E_f$ we have*

$$v \prec\!\!\prec_k w \Longleftrightarrow \sigma(v) \prec\!\!\prec_k \sigma(w),$$

$2n+1$	1	3	5	7	9	11	13	15	17	19	21
$\mathcal{P}_n$	1	1	2	7	32	175	1 083	7 342	53 372	409 982	3 293 148

23	25	27	29	31
27 446 089	235 943 180	2 082 554 573	18 804 608 658	173 194 661 758

33	35
1 623 164 580 385	15 448 388 973 479

Table 1: The numbers $\mathcal{P}_n$ of positive meanders with $2n+1$ nodes

Lemma 3.2 implies that for two order-equivalent attractors $\mathcal{A}_f, \mathcal{A}_g$ the corresponding permutations are equal:

$$\rho_f = \rho_g.$$

Due to a result of Fiedler and Rocha in [11] this implies even C^0-orbit equivalence of the attractors $\mathcal{A}_f$ and $\mathcal{A}_g$.

At the other hand results in [12] and [29] show that all meandric permutations with non-negative winding numbers, we call them *positive meanders*, can be realized as the permutation of the solutions to the stationary problem (2). Moreover, if the attractors $\mathcal{A}_f, \mathcal{A}_g$ belong to different equivalence classes, then Lemma 3.1 implies that their permutations have to be different. Together, this yields the following theorem:

Theorem 5.2 *There is a one to one correspondence between positive meanders and the classes of order-equivalent attractors.*

We want to discuss now this concept of order-equivalence. The first concept of equivalence for attractors of equation (1) has been introduced by Fiedler and Rocha in [10]: Two attractors were called connection-equivalent, if there is a bijection of the equilibria, preserving Morse-indices and connections. They showed also that connection-equivalence can be checked from the permutation, i.e. relies only on ODE-information about the solutions to the stationary problem. But already in [10] there were rather simple examples where connection-equivalence failed to give a satisfactory characterisation of the flow on the attractor. Completely different permutations turned out to have connection-equivalent attractors.

This difficulty was resolved in [28]. It has been pointed out that taking into account the sequence of strong-stable and strong-unstable manifolds according to the Sturm-Liouville spectra (cf. Proposition 3.3) allows for a more detailed characterisation of the attractors. Note that this structure is not regarded by C^0-orbit equivalence. Due to proposition 3.4 the distribution of the connecting orbits $C(v, w)$ among these manifolds is governed by the zero number $z(v - w)$. This gave rise to the following definition (see [28]):

Definition 5.3 *Two attractors are called Sturm-equivalent, if there is a bijection of the equilibria, preserving Morse-indices, connections, and zero-numbers* $z(v-w)$ *for connected equilibria* $v \searrow w$.

Lemma 5.4 *If two attractors* $\mathcal{A}_f$ *and* $\mathcal{A}_g$ *are order-equivalent, then they are also Sturm-equivalent.*

Proof: Due to Theorem 5.2, two order-equivalent attractors have the same permutation of the equilibria. This permutation determines all Morse-indices and zero-numbers (see Section 3), as well as the heteroclinic connections (see Lemma 3.1 and Theorem 2.4). □

The inverse statement, however, is in general not true. There exist Sturm-equivalent attractors where the corresponding permutations are different, and hence order-equivalence fails. An easy way to obtain such examples was shown in [10]. The transformation $T_1 : u \longmapsto -u$ in (1) leads simply to a symmetric image of the attractor. It is not difficult to figure out that the corresponding permutation ρ will be conjugated $T_1 : \rho \longmapsto \tau\rho\tau^{-1}$ where τ is the involution

$$\tau = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix}$$

Geometrically, this means a rotation of the meandric curve by 180 degrees. Obviously, the meandric curve $\tau\rho\tau^{-1}$ may differ from ρ. Another transformation $T_2 : x \longmapsto 1-x$ in (1), reflecting all x-profiles, acts on the permutation by $T_2 : \rho \longmapsto \rho^{-1}$. The actions of these transformations on the order-chains are the following:

$$v \lll_k w \Longleftrightarrow T_1(w) \lll_k T_1(v) \tag{11}$$

$$v \lll_k w \Longleftrightarrow \begin{cases} T_2(v) \lll_k T_2(w) \text{ and } k \text{ even} \\ T_2(w) \lll_k T_2(v) \text{ and } k \text{ odd} \end{cases} \tag{12}$$

Beside this two transformations, reflecting either all or only the odd order-chains, there exist further possibilities to obtain Sturm-equivalent but different permutations: For any subset $K = \{k_1, k_2, \dots\} \subseteq \mathbf{N}$, we may reflect the k-order-chains for all $k \in K$.

Example 5.5 *The permutations of 13 equilibria* $\rho_1 = (4\,12)(5\,11\,9)(6\,10\,8)$ *and* $\rho_2 = (2\,10)(3\,9\,7)(4\,8\,6)$ *are Sturm-equivalent. The corresponding bijection*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 1 & 12 & 11 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 13 \end{pmatrix}$$

reflects only the 0-order-chains, whereas the 1-order-chains and the 2-order-chains remain unchanged:

$$v \lll_k w \Longleftrightarrow \begin{cases} \sigma(v) \lll_k \sigma(w) \text{ and } k \geq 1 \\ \sigma(w) \lll_k \sigma(v) \text{ and } k = 0 \end{cases}$$

This may be generalized as follows: Due to a result in [24], the attractor can be parametrized globally by the the first m eigenfunctions $\phi_0, \ldots, \phi_{m-1}$, where m is the maximal unstable dimension of an equilibrium in the attractor. Note that the above mentioned reflection of the k-order-chains, $k \in K$, can be obtained by a transformation of the basis functions

$$\phi_k \longmapsto -\phi_k \quad \text{for all} \quad k \in K.$$

For a single k, the union of all k-order-chains may in general consist of several connected components (in the sense of the partial order). In such cases, each component can be reflected independently. But since the attractor is contained in a m-dimensional inertial manifold, a corresponding transformation of the attractor can not be extended to the inertial manifold. The simplest example, showing this phenomenon, are the following two permutations of 13 equilibria: $\rho_1 = (2\,4\,6)(3\,5)(8\,10\,12)(9\,11)$ and $\rho_2 = (2\,6\,4)(3\,5)(8\,10\,12)(9\,11)$.

In the case of order-equivalence, however, the orientation of the attractor in the span of the first m eigenfunctions is taken into account. This is reflected by the fact that in contrast to Sturm-equivalence, where for connected equilibria only the zero-number $z(v-w)$ is regarded, also the order of the values $v(0)$ and $w(0)$ enter into the fundamental notion

$$v \prec\!\!\prec_k w.$$

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