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Sufficient optimality conditions for the Moreau-Yosida-type regularization concept applied to semilinear elliptic optimal control problems with pointwise state constraints

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Abstract

We develop sufficient optimality conditions for a Moreau-Yosida regularized optimal control problem governed by a semilinear elliptic PDE with pointwise constraints on the state and the control. We make use of the equivalence of a setting of Moreau-Yosida regularization to a special setting of the virtual control concept, for which standard second order sufficient conditions have been shown. Moreover, we compare both regularization approaches within a numerical example.

1 Introduction

In this paper we consider the following class of semilinear optimal control problems with pointwise state and control constraints

$$\left.\begin{aligned} \min \quad J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \\ Ay + d(x,y) = u \quad \text{in } \Omega \\ \partial_{n_A} y = 0 \quad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. in } \Omega \\ y(x) \ge y_c(x) \quad \text{a.e. in } \bar{\Omega}. \end{aligned}\right\} \tag{P}$$

The precise conditions are given in Assumption 2.1. Due to the nonlinearity of the state equation the above model problem is of nonconvex type, which makes it necessary to consider sufficient optimality conditions ensuring local optimality of stationary points. We point out the results in [7, 8, 9] where second order sufficient conditions were established for semilinear elliptic control problems with pointwise state constraints. However, it is well known that Lagrange multipliers with respect to pointwise state constraints are in general only regular Borel measures, cf. [1, 4, 5]. The presence of these measures in the optimality system complicates the numerical treatment of such problems significantly, since a pointwise evaluation of the complementary slackness conditions is not possible. For that reason, several regularization concepts to overcome this lack of regularity have been developed in the recent past. We mention for example Lavrentiev regularization by Meyer, Rösch, and Tröltzsch, [20], the penalization method by Ito and Kunisch, [16], as well as interior point methods, cf. [27] and the references therein. Special methods have been developed for boundary control problems, such as an extension of Lavrentiev regularization by a source term representation of the control, see [30] and [23], and the virtual control approach [17]. This approach has been extended to distributed controls in [10] and turned out to be suitable for problems were control and state constraints are active simultaneously. Efficient optimization algorithms are available for all these regularized problems, see section 5 for detailed information. Concerning second order sufficient conditions for Lavrentiev regularized problems, we point out the results in [25]. For the Moreau-Yosida regularization concept, one can easily see that a classical second order analysis is not possible due to the fact that the regularized objective function is not twice differentiable.

However, by the equivalence of a specific settings of the Moreau-Yosida regularization and the virtual control concept we are able to derive a sufficient optimality condition for the Moreau-Yosida regularization making use of classical second order sufficient conditions for the virtual control concept. This condition ensures local optimality of controls satisfying the first order optimality conditions of Moreau-Yosida regularized problems. These results are not strictly limited to problem (P). In section 5, we therefore give examples of problem classes to which the theory can be extended, including boundary control problems as well as problems governed by parabolic PDEs.

2 Assumptions and properties of the state equation

We begin by briefly laying out the setting of the optimal control problem and stating some properties of the problem and the underlying PDE. Throughout the paper, we will use the following notation: By $\|\cdot\|$ we denote the usual norm in $L^2(\Omega)$, and $(\cdot,\cdot)$ is the associated inner product. The $L^\infty(\Omega)$-norm is specified by $\|\cdot\|_\infty$.

Assumption 2.1

- *The function $y_d \in L^2(\Omega)$ and $y_c \in L^\infty(\Omega)$ are given functions and $u_a < u_b$, $\nu > 0$ are real numbers.*
- *Ω denotes a bounded domain in $\mathbb{R}^N$, $N = \{2,3\}$, which is convex or has a $C^{1,1}$-boundary $\partial\Omega$.*
- *A denotes a second order elliptic operator of the form*

$$Ay(x) = -\sum_{i,j=1}^{d} \partial_{x_j}(a_{ij}(x)\partial_{x_i}y(x)),$$

where the coefficients a_{ij} belong to $C^{0,1}(\bar{\Omega})$ with the ellipticity condition

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \forall(x,\xi) \in \Omega \times \mathbb{R}^d, \quad \theta > 0.$$

Moreover, ∂_{n_A} denotes the conormal-derivative associated with A.

- *The function* $d = d(x,y)\colon \Omega\times\mathbb{R}$ *is measurable with respect to* $x \in \Omega$ *for all fixed* $y \in \mathbb{R}$*, and twice continuously differentiable with respect to* y*, for almost all* $x \in \Omega$*. Moreover,* d_{yy} *is assumed to be a locally bounded and locally Lipschitz continuous function with respect to* y*, i.e. the following Carathéodory type conditions hold true: there exists* $K > 0$ *such that*

  $$\|d(\cdot,0)\|_\infty + \|d_y(\cdot,0)\|_\infty + \|d_{yy}(\cdot,0)\|_\infty \le K$$

  and for any $M > 0$ *there exists* $L_M > 0$ *such that*

  $$\|d_{yy}(\cdot,y_1) - d_{yy}(\cdot,y_2)\|_\infty \le L_M|y_1 - y_2|$$

  for all $y_i \in \mathbb{R}$ *with* $|y_i| \le M$*,* $i = 1,2$*.*
  Additionally, we assume that $d_y(x,y)$ *is nonnegative for almost all* $x \in \Omega$ *and* $y \in \mathbb{R}$ *and positive on a set* $E_\Omega \times \mathbb{R}$*, where* $E_\Omega \subset \Omega$ *is of positive measure.*

Under the previous assumptions, we can deduce the following standard result for the state equation in problem (P):

Theorem 2.2 *Under Assumption 2.1 the semilinear elliptic boundary value problem*

$$\begin{aligned} Ay + d(x,y) &= u \quad \text{in } \Omega \\ \partial_{n_A} y &= 0 \quad \text{on } \Gamma \end{aligned} \tag{2.1}$$

admits for every right hand side $u \in L^2(\Omega)$ *a unique solution* $y \in H^1(\Omega) \cap C(\bar{\Omega})$*.*

The proof can be found e.g. in [6]. Based on this theorem, we introduce the control-to-state operator

$$G\colon L^2(\Omega) \to H^1(\Omega) \cap C(\bar{\Omega}),\ u \mapsto y, \tag{2.2}$$

that assigns to each $u \in L^2(\Omega)$ the weak solution $y \in H^1(\Omega) \cap C(\bar{\Omega})$ of (2.1). For future reference, we will provide results concerning differentiability of the control-to-state operator.

Theorem 2.3 *Let Assumption 2.1 be fulfilled. Then the mapping* $G\colon L^2(\Omega) \to H^1(\Omega) \cap C(\bar{\Omega})$*, defined by* $G(u) = y$ *is of class* C^2*. Moreover, for all* $u,\, h \in L^2(\Omega)$*,* $y_h = G'(u)h$ *is defined as the solution of*

$$\begin{aligned} Ay_h + d_y(x,y)y_h &= h \quad \text{in } \Omega \\ \partial_{n_A} y_h &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{2.3}$$

Furthermore, for every $h_1, h_2 \in L^2(\Omega)$*,* $y_{h_1,h_2} = G''(u)[h_1,h_2]$ *is the solution of*

$$\begin{aligned} Ay_{h_1,h_2} + d_y(x,y)y_{h_1,h_2} &= -d_{yy}(x,y)y_{h_1}y_{h_2} \quad \text{in } \Omega \\ \partial_{n_A} y_{h_1,h_2} &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{2.4}$$

where $y_{h_i} = G'(u)h_i$*,* $i = 1,2$*.*

Due to the convexity of the cost functional with respect to the control u and the associated state $y = G(u)$, the existence of at least one solution of problem (P) can be obtained by standard arguments, assuming that the set of feasible controls is nonempty. For future references, we define the set of admissible controls handling the box constraints

$$U_{ad} = \{u \in L^2(\Omega)\colon\ u_a \le u \le u_b \text{ a.e. in } \Omega\}. \tag{2.5}$$

Relying on the standard assumption of a Mangasarian-Fromovitz constraint qualification for the pure state constraints, we obtain the following first order necessary optimality conditions for a locally optimal control $\bar{u}$:

Theorem 2.4 *Let $\bar{u}$ be a solution of problem* (P) *and let $\bar{y} = G\bar{u}$ be the associated state. Then, a regular Borel measure $\bar{\mu} := \bar{\mu}_\Omega + \bar{\mu}_\Gamma \in \mathcal{M}(\bar{\Omega})$ and an adjoint state $\bar{p} \in W^{1,s}(\Omega)$, $s < d/(d-1)$ exist, such that the following optimality system is satisfied:*

$$\begin{aligned} A\bar{y} + d(x,\bar{y}) &= \bar{u} & \qquad A^*\bar{p} + d_y(x,\bar{y})\bar{p} &= \bar{y} - y_d - \bar{\mu}_\Omega \\ \partial_{n_A}\bar{y} &= 0 & \partial_{n_{A^*}}\bar{p} &= -\bar{\mu}_\Gamma \end{aligned} \tag{2.6}$$

$$(\bar{p} + \nu\bar{u}\,,\, u - \bar{u}) \ge 0, \quad \forall u \in U_{ad} \tag{2.7}$$

$$\begin{gathered} \int_{\bar{\Omega}} (y_c - \bar{y})d\bar{\mu} = 0, \quad \bar{y}(x) \ge y_c(x) \quad \text{for all } x \in \bar{\Omega} \\ \int_{\bar{\Omega}} \varphi d\bar{\mu} \ge 0 \quad \forall \varphi \in C(\bar{\Omega})^+, \end{gathered} \tag{2.8}$$

where $C(\bar{\Omega})^+$ is defined by $C(\bar{\Omega})^+ := \{y \in C(\bar{\Omega}) \,|\, y(x) \ge 0\, \forall\, x \in \bar{\Omega}\}$.

Here and in the following, A^* denotes the dual operator to the differential operator A. This result can be obtained adapting the theory of Casas, cf. [6].

With the help of the classical reduced Lagrange functional

$$\mathcal{L}(u,\mu) = J(G(u),u) + \int_{\bar{\Omega}} (y_c - G(u))\, d\mu,$$

the second order sufficient condition

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u},\bar{\mu})h^2 \ge \alpha\|h\|^2, \quad \alpha > 0,\ \forall h \in L^2(\Omega) \tag{2.9}$$

guarantees $\bar{u}$ to be a local minimum of (P) since the quadratic growth condition

$$J(G(u),u) \ge J(G(\bar{u}),\bar{u}) + \beta\|u - \bar{u}\|^2$$

is satisfied for a constant $\beta > 0$ for all $u \in U_{ad}$ in a sufficiently small L^2-neighborhood of $\bar{u}$.

3 Regularization approaches

The main focus of this paper is on regularized versions of problem (P). In this section we present the two regularization approaches we will examine in this paper, the Moreau-Yosida approximation on the one hand and the virtual control concept on the other. We will elaborate that the simple version of Moreau-Yosida regularization is equivalent to a special setting of the virtual control concept.

3.1 Moreau-Yosida regularization

The penalization technique by Ito and Kunisch, [16], based on a Moreau-Yosida approximation of the Lagrange multipliers with respect to the state constraints, applied to problem (P), leads to the following family of regularized problems

$$\left.\begin{aligned} \min \quad J^{MY}(y_\gamma, u_\gamma) := J(y_\gamma, u_\gamma) + \frac{\gamma}{2}\int_\Omega ((y_c - y_\gamma)_+)^2 dx \\ Ay_\gamma + d(x, y_\gamma) = u_\gamma \qquad \text{in } \Omega \\ \partial_{n_A} y_\gamma = 0 \qquad \text{on } \Gamma \\ u_a \le u_\gamma(x) \le u_b \quad \text{a.e. in } \Omega, \end{aligned}\right\} \tag{P^{MY}}$$

where $\gamma > 0$ is a regularization parameter that is taken large. Note, that the mapping $(\cdot)_+$ denotes the positive part of a measurable function, i.e. $(f)_+ = \max\{0, f\}$. Introducing a reduced formulation of problem (P^{MY}) by the control-to-state mapping G in (2.2) for the state equation, the following existence theorem can be proven since the set of admissible controls is nonempty.

Theorem 3.1 *Under Assumption 2.1, the regularized optimal control* (P^{MY}) *admits at least one (globally) optimal control $\bar{u}_\gamma$ with associated optimal state $\bar{y}_\gamma = G(\bar{u}_\gamma)$.*

Due to the nonlinearity of the state equation, the optimal control problem is nonconvex and one has to take into account the existence of multiple locally optimal controls. Forthcoming, let $\bar{u}_\gamma$ be a locally optimal control of problem (P^{MY}) with associated state $\bar{y}_\gamma = G(\bar{u}_\gamma)$. Using the classical Lagrange formulation, straight forward computations yield the following first order necessary optimality conditions.

Proposition 3.2 *Let $(\bar{y}_\gamma, \bar{u}_\gamma)$ be a locally optimal solution of problem* (P^{MY})*. Then, there exists a unique adjoint state $\bar{p}_\gamma \in H^1(\Omega) \cap C(\bar{\Omega})$ such that the following optimality system is satisfied*

$$\begin{aligned} A\bar{y}_\gamma + d(x, y_\gamma) &= \bar{u}_\gamma & \qquad A^*\bar{p}_\gamma + d_y(x, \bar{y}_\gamma)\bar{p}_\gamma &= \bar{y}_\gamma - y_d - \bar{\lambda}_\gamma \\ \partial_{n_A}\bar{y}_\gamma &= 0 & \partial_{n_{A^*}}\bar{p}_\gamma &= 0 \end{aligned} \tag{3.10}$$

$$(\bar{p}_\gamma + \nu\bar{u}_\gamma\, ,\, u - \bar{u}_\gamma) \ge 0 \quad \forall u \in U_{ad} \tag{3.11}$$

$$\bar{\lambda}_\gamma = \gamma(y_c - \bar{y}_\gamma)_+ \in L^2(\Omega) \tag{3.12}$$

Convergence analysis as γ tends to infinity is discussed in [21]. Convergence results of the Moreau-Yosida approximation applied to control and state constrained optimal control problems governed by semilinear parabolic PDEs are derived in [22].

3.2 Virtual control concept

In this section, we will apply the so called virtual control concept, first introduced in [17]. Instead of problem (P), we will investigate a family of regularized optimal control problems with mixed control-state constraints:

$$\left.\begin{aligned} \min \quad J^{VC}(y_\varepsilon, u_\varepsilon, v_\varepsilon) := J(y_\varepsilon, u_\varepsilon) + \frac{\psi(\varepsilon)}{2}\|v_\varepsilon\|^2_{L^2(\Omega)} \\ Ay_\varepsilon + d(x,y) = u_\varepsilon + \phi(\varepsilon)v_\varepsilon \quad \text{in } \Omega \\ \partial_{n_A} y_\varepsilon = 0 \quad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. in } \Omega \\ y_\varepsilon(x) \ge y_c(x) - \xi(\varepsilon)v_\varepsilon \quad \text{a.e. in } \Omega, \end{aligned}\right\} \tag{P^{VC}}$$

with a regularization parameter $\varepsilon > 0$ and positive and real valued parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$. The remaining given quantities are defined as for problem (P), see Assumption 2.1.

Analogously to the Moreau-Yosida approximation, the existence of at least one pair of optimal controls $(\bar{u}_\varepsilon,\, \bar{v}_\varepsilon)$ can be proven by standard arguments using a continuous control-to-state mapping due to the fact that $(\bar{u}, 0)$ is feasible for all problems (P^{VC}), where $\bar{u}$ denotes a locally optimal control of the original problem (P).
The existence of regular Lagrange multipliers with respect to mixed control-state constraints is known from e.g. [24] and [26]. Based on this, the following first order necessary optimality conditions for (P^{VC}) are obtained in a straight forward manner.

Proposition 3.3 *Let $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be an optimal solution of (P^{VC}) and let $\bar{y}_\varepsilon$ be the associated state. Then, there exist a unique adjoint state $\bar{p}_\varepsilon \in H^1(\Omega) \cap C(\bar{\Omega})$ and a unique Lagrange multiplier $\bar{\mu}_\varepsilon \in L^2(\Omega)$ so that the following optimality system is satisfied*

$$\begin{aligned} A\bar{y}_\varepsilon + d(x,\bar{y}_\varepsilon) &= \bar{u}_\varepsilon + \phi(\varepsilon)\bar{v}_\varepsilon & \qquad A^*\bar{p}_\varepsilon + d_y(x,\bar{y}_\varepsilon)\bar{p}_\varepsilon &= \bar{y}_\varepsilon - y_d - \bar{\mu}_\varepsilon \\ \partial_{n_A}\bar{y}_\varepsilon &= 0 & \partial_{n_{A^*}}\bar{p}_\varepsilon &= 0 \end{aligned} \tag{3.13}$$

$$(\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon \,,\, u - \bar{u}_\varepsilon) \ge 0, \quad \forall u \in U_{ad} \tag{3.14}$$

$$\phi(\varepsilon)\bar{p}_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon)\bar{\mu}_\varepsilon = 0, \quad \textit{a.e. in } \Omega \tag{3.15}$$

$$(\bar{\mu}_\varepsilon \,,\, y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon) = 0, \quad \bar{\mu}_\varepsilon \ge 0, \quad \bar{y}_\varepsilon \ge y_c - \xi(\varepsilon)\bar{v}_\varepsilon \quad \textit{a.e. in } \Omega. \tag{3.16}$$

The convergence of a sequence of regularized optimal controls $\bar{u}_\varepsilon$ to an optimal solution of the original problem (P) and the uniqueness of dual variables was discussed in [18].

3.3 Equivalence of the concepts

In this section, we will point out the equivalence of the Moreau-Yosida approximation to a special case of the virtual control concept. More precisely, we will demonstrate that the two optimal control problems admit the same optimal controls $\bar{u}_\varepsilon = \bar{u}_\gamma$ and we will then call the regularization concepts and the respective optimal control problems equivalent.

We observe the problems (P^{VC}) for the specific choice $\phi(\varepsilon) \equiv 0$, i.e.:

$$\left.\begin{aligned} \min \quad J^{VC}(y_\varepsilon, u_\varepsilon, v_\varepsilon) := J(y_\varepsilon, u_\varepsilon) + \frac{\psi(\varepsilon)}{2}\|v_\varepsilon\|^2_{L^2(\Omega)} \\ Ay_\varepsilon + d(x,y) = u_\varepsilon \quad \text{in } \Omega \\ \partial_{n_A} y_\varepsilon = 0 \quad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. in } \Omega \\ y_\varepsilon(x) \ge y_c(x) - \xi(\varepsilon) v_\varepsilon \quad \text{a.e. in } \Omega, \end{aligned}\right\} \qquad (\mathrm{Q}^{VC})$$

As one can easily see, there is no longer a coupling of both control variables by the state equation of the problem.

First, we consider both types of problems (Q^{VC}) and (P^{MY}) without any notice on the optimality conditions. We start investigating the mixed control-state constraints in (Q^{VC}) pointwise, where we split the domain Ω into two disjoint subsets $\Omega = \Omega_1 \cup \Omega_2$:

$$\begin{aligned} \Omega_1 &:= \{x \in \Omega : \, y_c(x) - y_\varepsilon(x) < 0 \text{ a.e. in } \Omega\} \\ \Omega_2 &:= \{x \in \Omega : \, y_c(x) - y_\varepsilon(x) \ge 0 \text{ a.e. in } \Omega\}. \end{aligned}$$

Initially, we consider Ω_1. The mixed constraints are given by $y_c(x) - y_\varepsilon(x) \le \xi(\varepsilon) v_\varepsilon(x)$ a.e. in Ω. Due to the minimization of the L^2-norm of the virtual control v_ε in the objective of (Q^{VC}), we derive

$$v_\varepsilon \equiv 0 \quad \text{a.e. in } \Omega_1.$$

Considering Ω_2, the inequality

$$\xi(\varepsilon) v_\varepsilon(x) \ge y_c(x) - y_\varepsilon(x) \ge 0$$

has to be satisfied. Choosing the virtual control as small as possible, we deduce

$$v_\varepsilon = \frac{1}{\xi(\varepsilon)}(y_c - y_\varepsilon) \quad \text{a.e. in } \Omega_2.$$

Concluding, the mixed control-state constraints can be replaced by the equation

$$v_\varepsilon = \frac{1}{\xi(\varepsilon)}(y_c - y_\varepsilon)_+.$$

Thus, the optimal control problem (Q^{VC}) can be rewritten equivalently in the form

$$\begin{aligned} \min \quad & J(y_\varepsilon, u_\varepsilon) + \frac{\psi(\varepsilon)}{2\xi(\varepsilon)^2}\|(y_c - y_\varepsilon)_+\|^2_{L^2(\Omega)} \\ & Ay_\varepsilon + d(x, y_\varepsilon) = u_\varepsilon \quad \text{in } \Omega \\ & \partial_{n_A} y_\varepsilon = 0 \quad \text{on } \Gamma \\ & u_a \le u_\varepsilon(x) \le u_b \quad \text{a.e. on } \Omega. \end{aligned}$$

Consequently, we formulate the following result.

Corollary 3.4 *For the specific parameter function $\phi(\varepsilon) \equiv 0$, the problem ($P^{VC}$) is equivalent to the problem (P^{MY}) arising by the Moreau-Yosida regularization, if the regularization parameter $\gamma > 0$ is defined by $\gamma := \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$.*

For the sake of completeness, we will additionally elaborate on the equivalence by the different first order necessary optimality conditions. Due to Proposition 3.3 and $\phi(\varepsilon) \equiv 0$, an optimal control $(\bar u_\varepsilon, \bar v_\varepsilon)$ of (Q^{VC}) satisfies

$$\begin{aligned} A\bar y_\varepsilon + d(x, \bar y_\varepsilon) &= \bar u_\varepsilon \qquad & A^*\bar p_\varepsilon + d_y(x, \bar y_\varepsilon)\bar p_\varepsilon &= \bar y_\varepsilon - y_d - \bar\mu_\varepsilon \\ \partial_{n_A}\bar y_\varepsilon &= 0 & \partial_{n_{A^*}}\bar p_\varepsilon &= 0 \end{aligned} \tag{3.17}$$

$$(\bar p_\varepsilon + \nu\bar u_\varepsilon\, ,\, u - \bar u_\varepsilon) \ge 0, \quad \forall u \in U_{ad} \tag{3.18}$$

$$\psi(\varepsilon)\bar v_\varepsilon - \xi(\varepsilon)\bar\mu_\varepsilon = 0, \quad \text{a.e. in } \Omega \tag{3.19}$$

$$(\bar\mu_\varepsilon\, ,\, y_c - \bar y_\varepsilon - \xi(\varepsilon)\bar v_\varepsilon) = 0, \quad \bar\mu_\varepsilon \ge 0, \quad \bar y_\varepsilon \ge y_c - \xi(\varepsilon)\bar v_\varepsilon \quad \text{a.e. in } \Omega \tag{3.20}$$

Since the multiplier $\bar\mu_\varepsilon$ is a regular function, it is well known that the complementary slackness conditions in (3.20) are equivalent to

$$\bar\mu_\varepsilon - \max\{0, \bar\mu_\varepsilon + c(y_c - \bar y_\varepsilon - \xi(\varepsilon)\bar v_\varepsilon)\} = 0$$

for every $c > 0$. Using the specific choice $c = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$, we obtain

$$\bar\mu_\varepsilon = \max\{0, \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(y_c - \bar y_\varepsilon)\} = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(y_c - \bar y_\varepsilon)_+.$$

instead of (3.19) and (3.20). Due to (3.19), the virtual control satisfies

$$\bar v_\varepsilon = \frac{\xi(\varepsilon)}{\psi(\varepsilon)}\bar\mu_\varepsilon = \frac{1}{\xi(\varepsilon)}(y_c - \bar y_\varepsilon)_+. \tag{3.21}$$

By means of Proposition 3.2, it is easily seen that the optimality systems of (P^{MY}) and (Q^{VC}) are equivalent and we conclude with the following result.

Corollary 3.5 *Let $(\bar y_\varepsilon, \bar u_\varepsilon, \bar v_\varepsilon)$ be a stationary point of (P^{VC}). If we set $\phi(\varepsilon) \equiv 0$, then the virtual control can be represented by $\bar v_\varepsilon = 1/\xi(\varepsilon)(y_c - \bar y_\varepsilon)_+$. Moreover, $(\bar y_\varepsilon, \bar u_\varepsilon)$ is also a stationary point of (P^{MY}) for the specific choice $\gamma = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$. Conversely, a stationary point of (P^{MY}) is also a stationary point of (P^{VC}) if the conditions above are satisfied.*

4 Sufficient optimality conditions for the Moreau-Yosida approximation

Now we will formulate a sufficient optimality condition for the Moreau-Yosida approximation based on a second order sufficient optimality condition for the respective equivalent virtual control concept (Q^{VC}). We first define the Lagrangian of problem (Q^{VC}) by

$$\begin{aligned}\mathcal{L}^{VC}(u,v,\mu) = \frac{1}{2}\|G(u)-y_d\|^2 + \frac{\nu}{2}\|u\|^2 + \frac{\psi(\varepsilon)}{2}\|v\|^2 \\ + \int_\Omega (y_c - G(u) - \xi(\varepsilon)v)\mu\, dx\end{aligned} \tag{4.22}$$

using the control-to-state operator G, given in (2.2). Straight forward computations show that the second derivative of the Lagrangian is given by

$$\begin{aligned}\frac{\partial^2\mathcal{L}^{VC}(u,v,\mu)}{\partial(u,v)^2}[h_1,h_2] =& (G'(u)h_{u,1}, G'(u)h_{u,2}) + (G(u)-y_d, G''(u)[h_{u,1},h_{u,2}]) \\ &+ \nu(h_{u,1},h_{u,2}) + \psi(\varepsilon)(h_{v,1},h_{v,2}) - (G''(u)[h_{u,1},h_{u,2}],\mu)\end{aligned} \tag{4.23}$$

for $h_i = (h_{u,i}, h_{v,i}) \in L^2(\Omega)^2$, $i = 1,2$. In the sequel, let $(\bar{u}_\varepsilon,\, \bar{v}_\varepsilon)$ be a local solution of (Q^{VC}) with associated Lagrange multiplier $\bar{\mu}_\varepsilon$, i.e. (3.17)-(3.20) are satisfied. We proceed with establishing the second order sufficient optimality condition.

Assumption 4.1 *There exists a constant $\alpha \geq 0$ such that*

$$\frac{\partial^2\mathcal{L}^{VC}(\bar{u}_\varepsilon,\bar{v}_\varepsilon,\bar{\mu}_\varepsilon)}{\partial(u,v)^2}[h_u,h_v]^2 \geq \alpha\|h_u\|^2 + \psi(\varepsilon)\|h_v\|^2 \tag{4.24}$$

is valid for all $h_u \in L^2(\Omega)$.

Note, that the coercivity condition of the second derivative of the Lagrangian with respect to directions $h_v \in L^2(\Omega)$ is satisfied by construction, see (4.23). Based on the previous coercivity condition, one can prove a quadratic growth condition for problem (Q^{VC}) that ensures local optimality of $(\bar{u}_\varepsilon,\, \bar{v}_\varepsilon)$.

Proposition 4.2 *Let $(\bar{u}_\varepsilon,\, \bar{v}_\varepsilon)$ be a control satisfying the first order necessary optimality conditions* (3.17)-(3.20)*. Additionally, $(\bar{u}_\varepsilon,\, \bar{v}_\varepsilon)$ fulfills Assumption 4.1. Then, there exist constants $\beta > 0$ and $\delta > 0$ such that*

$$J^{VC}(G(u),u,v) \geq J^{VC}(G(\bar{u}_\varepsilon),\bar{u}_\varepsilon,\bar{v}_\varepsilon) + \beta(\|u-\bar{u}_\varepsilon\|^2 + \|v-\bar{v}_\varepsilon\|^2) \tag{4.25}$$

for all feasible controls $(u,\, v)$ of problem (Q^{VC}) with $\|u-\bar{u}_\varepsilon\| \leq \delta$.

Proof. First, let us mention that there is a specific difference to the standard proofs, since no smallness condition for $\|v - \bar{v}_\varepsilon\|$ is required. Let $(u, v) \in U_{ad} \times L^2(\Omega)$ be an admissible control of problem (Q^{VC}), i.e. mainly $y_c - \xi(\varepsilon)v - G(u) \leq 0$. In view of the positivity of the optimal Lagrange multiplier $\bar{\mu}_\varepsilon$, we can estimate the cost functional J^{VC} by the Lagrange functional:

$$J^{VC}(G(u), u, v) \geq J^{VC}(G(u), u, v) + \int_\Omega (y_c - G(u) - \xi(\varepsilon)v)\bar{\mu}_\varepsilon \, dx = \mathcal{L}(u, v, \bar{\mu}_\varepsilon).$$

Under Assumption 2.1, the Lagrange functional is twice continuously differentiable with respect to the $L^2(\Omega)$-norms, since the solution operator G has this property, see Theorem 2.3. Then, a Taylor expansion is given by

$$\begin{aligned}\mathcal{L}^{VC}(u, v, \bar{\mu}_\varepsilon) = \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon) + \frac{\partial \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)}{\partial(u, v)}(u - \bar{u}_\varepsilon, v - \bar{v}_\varepsilon)\\ + \frac{1}{2}\frac{\partial^2 \mathcal{L}^{VC}(\tilde{u}, \tilde{v}, \bar{\mu}_\varepsilon)}{\partial(u, v)^2}(u - \bar{u}_\varepsilon, v - \bar{v}_\varepsilon)^2\end{aligned}$$

with $\tilde{u} = \bar{u}_\varepsilon + \theta(u - \bar{u}_\varepsilon)$, $\tilde{v} = \bar{v}_\varepsilon + \theta(v - \bar{v}_\varepsilon)$ for a $\theta \in (0, 1)$. Since $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ satisfies the first order necessary optimality conditions (3.17)-(3.20) and $\bar{\mu}_\varepsilon$ is the associated Lagrange multiplier, we have

$$\frac{\partial \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)}{\partial(u, v)}(u - \bar{u}_\varepsilon, v - \bar{v}_\varepsilon) \geq 0 \quad \text{and} \quad \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon) = J^{VC}(G(\bar{u}_\varepsilon), \bar{u}_\varepsilon, \bar{v}_\varepsilon),$$

which implies

$$\begin{aligned}\mathcal{L}^{VC}(u, v, \bar{\mu}_\varepsilon) \geq J^{VC}(G(\bar{u}_\varepsilon), \bar{u}_\varepsilon, \bar{v}_\varepsilon) + \frac{1}{2}\frac{\partial^2 \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)}{\partial(u, v)^2}(u - \bar{u}_\varepsilon, v - \bar{v}_\varepsilon)^2\\ + \frac{1}{2}\left(\frac{\partial^2 \mathcal{L}^{VC}(\tilde{u}, \tilde{v}, \bar{\mu}_\varepsilon)}{\partial(u, v)^2} - \frac{\partial^2 \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)}{\partial(u, v)^2}\right)(u - \bar{u}_\varepsilon, v - \bar{v}_\varepsilon)^2.\end{aligned}$$

Using the SSC of Assumption 4.1, we obtain

$$\begin{aligned}\mathcal{L}^{VC}(u, v, \bar{\mu}_\varepsilon) \geq J^{VC}(G(\bar{u}_\varepsilon), \bar{u}_\varepsilon, \bar{v}_\varepsilon) + \alpha\|u - \bar{u}_\varepsilon\|^2 + \psi(\varepsilon)\|v - \bar{v}_\varepsilon\|^2\\ + \frac{1}{2}\left(\frac{\partial^2 \mathcal{L}^{VC}(\tilde{u}, \tilde{v}, \bar{\mu}_\varepsilon)}{\partial(u, v)^2} - \frac{\partial^2 \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)}{\partial(u, v)^2}\right)(u - \bar{u}_\varepsilon, v - \bar{v}_\varepsilon)^2.\end{aligned}$$

One can easily see that the second derivative (4.23) is independent of the virtual control v since the control-to-state operator is only applied to the control variable u and linear mixed control-state constraints are considered. Moreover, one can prove under Assumption 2.1 that the second derivative of the Lagrangian (4.23) is locally Lipschitz continuous with respect to u, i.e. there exists a positive constant C_L such that the estimate

$$\left|\left(\frac{\partial^2 \mathcal{L}^{VC}(u_1, v, \mu)}{\partial(u, v)^2} - \frac{\partial^2 \mathcal{L}^{VC}(u_2, v, \mu)}{\partial(u, v)^2}\right)h^2\right| \leq C_L\|u_1 - u_2\|\|h\|^2$$

holds true for $\|u_1 - u_2\| \leq \delta$ and $\delta > 0$ sufficiently small, see for instance [29, Lemma 4.24]. By means of the Lipschitz property concerning u and the independency of v, see (4.23), we conclude

$$\begin{aligned}\mathcal{L}^{VC}(u, v, \bar{\mu}_\varepsilon) &\geq J^{VC}(G(\bar{u}_\varepsilon), \bar{u}_\varepsilon, \bar{v}_\varepsilon) + \alpha\|u - \bar{u}_\varepsilon\|^2 + \psi(\varepsilon)\|v - \bar{v}_\varepsilon\|^2 \\ &\quad - c\|u - \bar{u}_\varepsilon\|(\|u - \bar{u}_\varepsilon\|^2 + \|v - \bar{v}_\varepsilon\|^2) \\ &\geq J^{VC}(G(\bar{u}_\varepsilon), \bar{u}_\varepsilon, \bar{v}_\varepsilon) + (\alpha - c\delta)\|u - \bar{u}_\varepsilon\|^2 + (\psi(\varepsilon) - c\delta)\|v - \bar{v}_\varepsilon\|^2,\end{aligned}$$

provided that $\|u - \bar{u}_\varepsilon\| \leq \delta$. For sufficiently small $\delta > 0$, we find a positive constant $\beta > 0$ such that the assertion is fulfilled. □

Forthcoming, we will rewrite the second order sufficient optimality condition of problem (Q^{VC}) in terms of the equivalent Moreau-Yosida regularization (P^{MY}) using relations between the respective variables derived in the previous section.

Due to Corollary 3.5, the control $\bar{u}_\varepsilon$ satisfies the first order optimality conditions (3.10)-(3.12) of (P^{MY}) with $\gamma = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$. Thus, we set

$$\bar{u}_\lambda = \bar{u}_\varepsilon, \quad \bar{\lambda}_\gamma = \bar{\mu}_\varepsilon = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(y_c - \bar{y}_\varepsilon)_+$$

and the SSC (4.24) of Assumption 4.1 yields the following

$$\|G'(\bar{u}_\gamma)h_u\|^2 + \nu\|h_u\|^2 + (G(\bar{u}_\gamma) - y_d, G''(\bar{u}_\gamma)h_u^2) - (\bar{\lambda}_\gamma, G''(\bar{u}_\gamma)h_u^2) \geq \alpha\|h_u\|^2 \quad (4.26)$$

for all $h_u \in L^2(\Omega)$, written in terms of the Moreau-Yosida regularization. Summarizing, one ends up with

$$J''(G(\bar{u}_\gamma), \bar{u}_\gamma)h_u^2 - (\bar{\lambda}_\gamma, G''(\bar{u}_\gamma)h_u^2) \geq \alpha\|h_u\|^2.$$

Concluding, we can state the following result.

Theorem 4.3 *Let $\bar{u}_\gamma \in U_{ad}$, with associated state $\bar{y}_\gamma = G(\bar{u}_\gamma)$, be a control satisfying the first order necessary optimality* (3.10)-(3.12). *Additionally, there exists a constant $\alpha > 0$ such that*

$$J''(G(\bar{u}_\gamma), \bar{u}_\gamma)h_u^2 - \gamma((y_c - G(\bar{u}_\gamma))_+, G''(\bar{u}_\gamma)h_u^2) \geq \alpha\|h_u\|^2 \quad (4.27)$$

is fulfilled for all $h_u \in L^2(\Omega)$. Then, there exist constants $\beta > 0$ and $\delta > 0$ so that

$$J^{MY}(G(u_\gamma), u_\gamma) \geq J^{MY}(G(\bar{u}_\gamma), \bar{u}_\gamma) + \beta\|u_\gamma - \bar{u}_\gamma\|^2 \quad (4.28)$$

holds for all $u_\gamma \in U_{ad}$ with $\|u_\gamma - \bar{u}_\gamma\| \leq \delta$. In particular, $(G(\bar{u}_\gamma), \bar{u}_\gamma)$ is a locally optimal solution of (P^{MY}).

Proof. Due to Corollary 3.5 and (3.21), the pair ($\bar{u}_\gamma$, $\bar{v}_\gamma := \frac{1}{\xi(\varepsilon)}(y_c - \bar{y}_\gamma)_+$) satisfies the first order optimality conditions (3.17)-(3.20) of problem (Q^{VC}), where the parameter functions $\psi(\varepsilon)$ and $\xi(\varepsilon)$ are chosen in a way such that $\gamma = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$. The associated Lagrange multiplier in the optimality conditions is denoted by $\bar{\mu}_\gamma$. Due to the former argumentation, one can easily see, that (4.27) implies the coercivity condition (4.24) in the point $(\bar{u}_\gamma,\, \bar{v}_\gamma,\, \bar{\mu}_\gamma)$, i.e.

$$\frac{\partial^2 \mathcal{L}^{VC}(\bar{u}_\gamma,\, \bar{v}_\gamma,\, \bar{\mu}_\gamma)}{\partial(u,v)^2}[h_u, h_v] \geq \alpha\|h_u\|^2 + \psi(\varepsilon)\|h_v\|^2$$

for all $h_u \in L^2(\Omega)$. Thus, Assumption 4.1 is satisfied and we proceed by applying Proposition 4.25. Hence, there exist constants $\beta > 0$ and $\delta > 0$ such that

$$J^{VC}(G(u), u, v) \geq J^{VC}(G(\bar{u}_\gamma),\, \bar{u}_\gamma,\, \bar{v}_\gamma) + \beta(\|u - \bar{u}_\gamma\|^2 + \|v - \bar{v}_\gamma\|^2)$$

for all feasible $(u,\, v)$ of problem (Q^{VC}) with $\|u - \bar{u}_\gamma\| \leq \delta$. Now, we consider an arbitrary control $u \in U_{ad}$ with $\|u - \bar{u}_\gamma\| \leq \delta$. Furthermore, the pair of controls $(u,\, v := \frac{1}{\xi(\varepsilon)}(y_c - G(u))_+)$ is feasible for problem (Q^{VC}) since

$$\xi(\varepsilon)v = (y_c - G(u))_+ \geq y_c - G(u).$$

By means of the equivalence of the problems (P^{MY}) and (Q^{VC}) and $\gamma = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$, we deduce

$$J^{MY}(G(\bar{u}_\gamma), \bar{u}_\gamma) = J^{VC}(G(\bar{u}_\gamma), \bar{u}_\gamma\, \bar{v}_\gamma) \quad \text{and} \quad J^{MY}(G(u), u) = J^{VC}(G(u), u, v).$$

Concluding, we obtain the assertion

$$J^{MY}(G(u), u) \geq J^{MY}(G(\bar{u}_\gamma), \bar{u}_\gamma) + \beta\|u - \bar{u}_\gamma\|^2$$

for all $u \in U_{ad}$ with $\|u - \bar{u}_\gamma\| \leq \delta$. □

Referring to [18], we point out that for sufficiently small ε the second order sufficient condition (4.1) can be deduced from the second order sufficient condition (2.9) of the original problem (P). This was proven under the assumption that the dual variables $\bar{\mu}$ and $\bar{p}$ introduced in Theorem 2.4 are unique. By the previously shown equivalence of the two regularization concepts, it is clear that under similar assumptions the sufficient condition (4.27) can also be derived from (2.9).

Corollary 4.4 *Let $\bar{u}$ fulfill the first order necessary optimality conditions of Theorem 2.4 with unique dual variables $\bar{\mu}$ and $\bar{p}$, as well as the second order sufficient condition* (2.9). *Then there exists a constant $\alpha > 0$ such that*

$$J''(G(\bar{u}_\gamma), \bar{u}_\gamma)h_u^2 - \gamma((y_c - G(\bar{u}_\gamma))_+, G''(\bar{u}_\gamma)h_u^2) \geq \alpha\|h_u\|^2$$

is fulfilled for all $h_u \in L^2(\Omega)$ provided that γ is sufficiently large.

5 Generalizations

In this section we want to point out that the theory presented in this paper can be generalized to large classes of semilinear optimal control problems. Let us start with an elliptic boundary control problem. The virtual control formulation with $\phi(\varepsilon) = 0$ is given by

$$\left.\begin{aligned} \min \quad J(y_\varepsilon, u_\varepsilon, v_\varepsilon) := \frac{\alpha_1}{2}\|y_\varepsilon - y_{d,\Omega}\|^2_{L^2(\Omega)} + \frac{\alpha_2}{2}\|y_\varepsilon - y_{d,\Gamma}\|^2_{L^2(\Gamma)} \\ +\frac{\nu}{2}\|u_\varepsilon\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|v_\varepsilon\|^2_{L^2(\Omega)} \\ Ay_\varepsilon + d(x, y_\varepsilon) = 0 \quad \text{in } \Omega \\ \partial_{n_A} y_\varepsilon + b(x, y_\varepsilon) = u_\varepsilon \quad \text{on } \Gamma \\ u_a \le u_\varepsilon(x) \le u_b \quad \text{a.e. in } \Gamma \\ y_\varepsilon(x) \ge y_c(x) - \xi(\varepsilon) v_\varepsilon \quad \text{a.e. in } \Omega, \end{aligned}\right\} \qquad (\mathrm{Q}_1^{VC})$$

and the corresponding equivalent Moreau-Yosida regularization is presented by

$$\left.\begin{aligned} \min \quad J(y_\gamma, u_\gamma, v_\gamma) := \frac{\alpha_1}{2}\|y_\gamma - y_{d,\Omega}\|^2_{L^2(\Omega)} + \frac{\alpha_2}{2}\|y_\gamma - y_{d,\Gamma}\|^2_{L^2(\Gamma)} \\ +\frac{\nu}{2}\|u_\gamma\|^2_{L^2(\Gamma)} + \frac{\gamma}{2}\|(y_c - y_\gamma)_+\|^2_{L^2(\Omega)} \\ Ay_\gamma + d(x, y_\gamma) = 0 \quad \text{in } \Omega \\ \partial_{n_A} y_\gamma + b(x, y_\gamma) = u_\gamma \quad \text{on } \Gamma \\ u_a \le u_\gamma(x) \le u_b \quad \text{a.e. in } \Gamma. \end{aligned}\right\} \qquad (\mathrm{P}_1^{MY})$$

The theory presented in section 3 can be adapted by only changing the corresponding sets. The results of section 4 depend on the dimension of the domain. For dimension $N = 3$ we get a two norm discrepancy in the second order sufficient optimality condition of proposition 4.2 in the virtual control approach, but only for the original control u. Of course, the corresponding sufficient optimality condition for the Moreau-Yosida regularization in theorem 4.3 contains a two norm setting, too. Let us mention that in this case sufficient optimality conditions for the unregularized problems are challenging due to regularity problems. Therefore, corollary 4.4 is then not verified by our theory.

It is also possible to generalize the theory to the regularized version of parabolic optimal control problems like

$$\left.\begin{aligned} \min \quad J(y, u) := \frac{\alpha_1}{2}\|y - y_d\|^2_{L^2(Q)} + \frac{\alpha_2}{2}\|y(T) - y_T\|^2_{L^2(\Omega)} \\ +\frac{\alpha_3}{2}\|y - y_\Sigma\|^2_{L^2(\Sigma)} + \frac{\nu}{2}\|u\|^2_{L^2(Q)} \\ y_t + Ay + d(t, x, y) = u \quad \text{in } Q = (0, T) \times \Omega \\ \partial_{n_A} y + b(t, x, y) = 0 \quad \text{on } \Sigma = (0, T) \times \Gamma \\ u_a \le u(t, x) \le u_b \quad \text{a.e. in } Q \\ y(t, x) \ge y_c(t, x) \quad \text{a.e. in } Q, \\ y(0) = y_0. \end{aligned}\right\} \qquad (\mathrm{P}_2)$$

Due to the weaker differentiability properties of parabolic control-to-state operators, a two norm discrepancy will have to be taken into account in proposition 4.2 and theorem 4.3 for spatial dimensions greater than one if $b \equiv 0$, and regardless of the spatial dimension if $b \not\equiv 0$. Similarly to the elliptic problem, corollary 4.4 is then not verified.

Moreover, it is possible to discuss more general objectives and nonlinearities in the partial differential equations with respect to the control u. However, then the discussion of the differentiability of the control-to-state mapping becomes more involved. In addition, one needs several technical assumptions on the nonlinearities to get the desired results. Such assumptions are essentially that ones that were needed for the derivation of sufficient second order conditions, see [25]. These discussions go beyond the scope of the paper.

6 Numerical example

In this section we will compare both of the previously presented the regularization approaches numerically. Therefore, we construct an optimal solution for the following optimal control problem

$$\left.\begin{aligned} \min \quad J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \\ \Delta y + y + y^3 = u + f \quad \text{in } \Omega \\ \partial_n y = 0 \quad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. in } \Omega \\ y(x) \ge y_c(x) \quad \text{a.e. in } \bar{\Omega}, \end{aligned}\right\} \tag{PT}$$

with $\Omega = [0,1]^2$ denotes the unit square. The Tikhonov regularization parameter was set to $\nu = 1 \cdot 10^{-3}$. It is well known that Lagrange multipliers associated to pointwise state constraints are in general only regular Borel measures, see e.g. [3] or [4]. In order to construct an analytical solution $(\bar{u}, \bar{y})$, we have to satisfy the optimality system

$$\begin{aligned} \Delta\bar{y} + \bar{y} + \bar{y}^3 &= \bar{u} + f & \qquad \Delta p + p + 3\bar{y}^2 p &= \bar{y} - y_d - \mu \\ \partial_n \bar{y} &= 0 & \partial_n p &= 0 \end{aligned}$$

$$(p + \nu\bar{u}\,,\, u - \bar{u}) \ge 0, \quad \forall u \in U_{ad}$$

$$\int_{\bar{\Omega}} (y_c - \bar{y})d\mu = 0, \quad \bar{y}(x) \ge y_c(x) \quad \text{for all } x \in \bar{\Omega}$$

$$\int_{\bar{\Omega}} \varphi d\mu \ge 0 \quad \forall \varphi \in C(\bar{\Omega})^+,$$

with an adjoint state p and a Lagrange Multiplier μ, see [6]. For the optimal state, control and adjoint state we choose

$$\bar{y}(x) = -16x_1^4 + 32x_1^3 - 16x_1^2 + 1,\; p(x) = 2x_1^3 - 3x_1^2,\; \bar{u}(x) = \Pi_{[u_a,u_b]}\{-\frac{p(x)}{\nu}\},$$

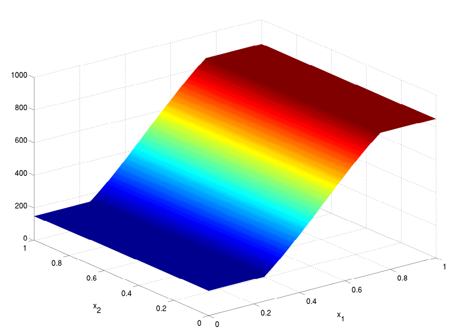

Figure 1: Control u_γ

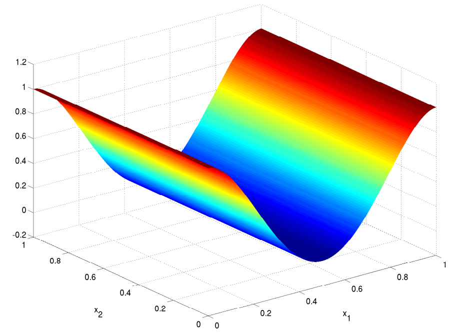

Figure 2: State y_γ

with $u_a = 150$ and $u_b = 850$, such that the gradient equation in the previous optimality system is fulfilled. Moreover, one can easily check that the homogeneous Neumann boundary conditions in the state and the adjoint equation are satisfied. The lower bound y_c is defined by

$$y_c(x) = \min\{\bar{y}(x_1 = 0.2), \bar{y}(x)\}.$$

This implies that the state constraints are active in $\Omega_a = \{x \in \Omega\colon 0.2 \leq x_1 \leq 0.8\}$. Continuatively, with

$$\mu(x) = \max\{0, \bar{y}(x_1 = 0.2) - \bar{y}(x)\}$$

the complementary slackness condition in the optimality system is satisfied. Finally, the partial differential equations of the optimality system yield

$$\begin{aligned} f &= -\Delta\bar{y} + \bar{y} + \bar{y}^3 - \bar{u} \\ y_d &= \Delta p - p - 3\bar{y}^2 p + \bar{y} - \mu. \end{aligned}$$

Notice, that the active sets associated to the pure state constraints and active set corresponding to the control constraints are not disjoint. Thus, a regularization of the problem by the virtual control approach is reasonable.

Forthcoming, the test problem (PT) is regularized by the Moreau-Yosida regularization and by the virtual control approach, see Section 3. The particular problems are denoted by (PT^{MY}) and (PT^{VC}), respectively. These optimization problems were solved numerically by a SQP method that is described in detail for instance in [15] and [28]. Furthermore, a primal-dual active set strategy is used solving the arised linear quadratic subproblems, see e.g. [2, 11, 12, 19] and the references therein. All functions were discretized by piecewise linear ansatz functions, defined on a uniform finite element mesh. The number of intervals in one dimension, denoted by N, is related to the mesh size by $h = \sqrt{2}N$. In the following all computations were performed with $N = 192$.

The Figures 1-4 show the numerical solution of the Moreau-Yosida approximation

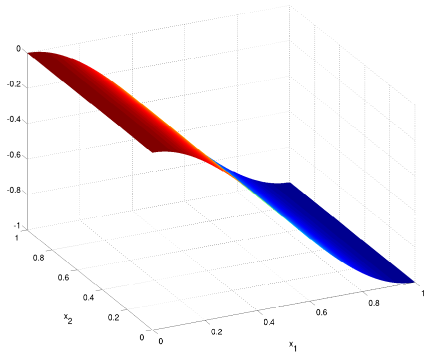

Figure 3: Adjoint state p_γ

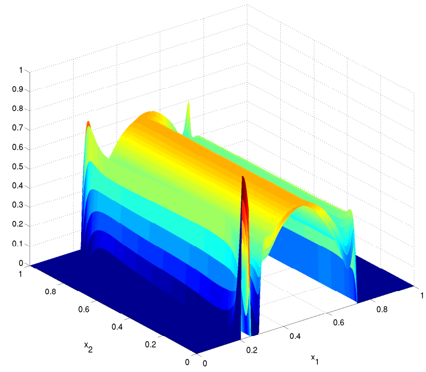

Figure 4: Approximation of Lagrange multiplier λ_γ

of problem (PT) for the fixed penalization parameter $\lambda = 1 \cdot 10^5$. In Figure 4 one can see irregularities on the boundary and in the parts of the domain, where the active sets of the original problem (PT) associated to the different constraints are not disjoint. Due to the knowledge of an analytical solution of problem (PT), we obtain the following error of the numerical solution of problem (PT^{MY}):

$$\|u_\gamma - \bar{u}\| \approx 3.1426e{-}02, \quad \|y_\gamma - \bar{y}\| \approx 2.7497e{-}05, \quad \|p_\gamma - p\| \approx 1.5147e{-}04. \tag{6.29}$$

The convergence behavior of the SQP method is presented in Table 1. We displayed the value of the cost functional J^{MY} for each step of SQP algorithm as well as the relative difference between two iterates, that is defined by

$$\delta_\gamma = \frac{1}{3}\left(\frac{\|u_\gamma^{(n)} - u_\gamma^{(n+1)}\|}{\|u_\gamma^{(n+1)}\|} + \frac{\|y_\gamma^{(n)} - y_\gamma^{(n+1)}\|}{\|y_\gamma^{(n+1)}\|} + \frac{\|p_\gamma^{(n)} - p_\gamma^{(n+1)}\|}{\|p_\gamma^{(n+1)}\|} \right).$$

Moreover, this quantity is used for a termination condition of the SQP method. In all numerical test the algorithm stops if $\delta < 1 \cdot 10^{-6}$. In addition the number of iterations of the primal-dual active set strategy is shown.

it_{SQP}	J^{MY}	δ_γ	$\#it_{AS}$
1	$1.497214e+02$	$1.414707e+00$	13
2	$1.766473e+02$	$3.585474e-01$	32
3	$1.767212e+02$	$4.605969e-02$	12
4	$1.767218e+02$	$7.115167e-04$	6
5	$1.767218e+02$	$2.087841e-07$	1

Table 1: Convergence of SQP-method for (PT^{MY})

We proceed with the virtual control approach (PT^{VC}) applied to the problem (PT). The numerical solution is denoted by the subscript ε. The parameter functions are chosen by

$$\psi(\varepsilon) \equiv 1, \quad \phi(\varepsilon) = \sqrt[4]{\varepsilon}, \quad \xi(\varepsilon) = \sqrt{\varepsilon}.$$

Hence, a regularization effect, which is similar to the Moreau-Yosida approximation above, is expected for $\varepsilon = 1 \cdot 10^5$. However, due to the choice $\phi(\varepsilon) \neq 0$, (PT^{VC}) is not equivalent to the Moreau-Yosida approximation of problem. Figures 5 and 6 shows the optimal virtual control v_ε of problem (PT^{VC}) and the Lagrange multiplier associated to the mixed control-state constraints. As one can see, the virtual control is compared to the Lagrange multiplier less irregular in the parts of the domain, where both kind of constraints are active. Except minor differences, the Lagrange multiplier μ_ε in Figure 6 and λ_γ in Figure 4 from the Moreau-Yosida approximation have an identical shape. Note, that we skipped displaying the optimal control u_ε,

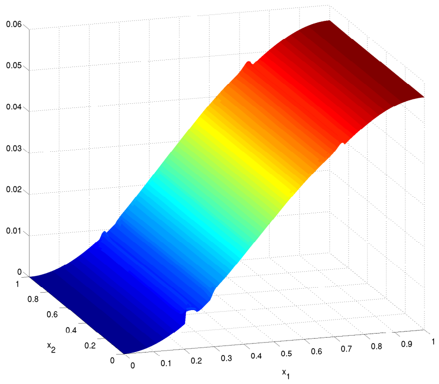

Figure 5: Control v_ε

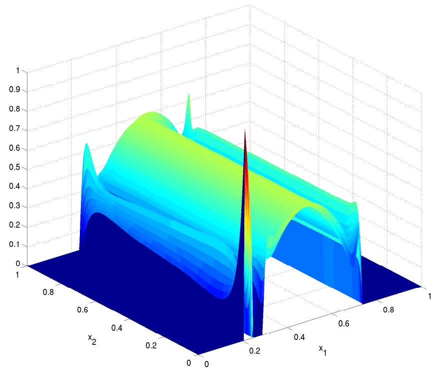

Figure 6: Lagrange multiplier μ_ε

the state y_ε and the adjoint state p_ε, since there is no visible difference to the Figures 1-3 recognizable. The convergence of the SQP method is illustrated in Table 2. The quantity δ_ε for the termination condition is now defined by

$$\delta_\varepsilon = \frac{1}{4}\left(\frac{\|u_\varepsilon^{(n)} - u_\varepsilon^{(n+1)}\|}{\|u_\varepsilon^{(n+1)}\|} + \frac{\|v_\varepsilon^{(n)} - v_\varepsilon^{(n+1)}\|}{\|v_\varepsilon^{(n+1)}\|} + \frac{\|y_\varepsilon^{(n)} - y_\varepsilon^{(n+1)}\|}{\|y_\varepsilon^{(n+1)}\|} + \frac{\|p_\varepsilon^{(n)} - p_\varepsilon^{(n+1)}\|}{\|p_\varepsilon^{(n+1)}\|} \right).$$

it_{SQP}	J^{VC}	δ_ε	$\#it_{AS}$
1	$1.497193e+02$	$1.311076e+00$	14
2	$1.766467e+02$	$1.298795e+00$	32
3	$1.767206e+02$	$6.030644e-02$	11
4	$1.767212e+02$	$3.294671e-03$	6
5	$1.767212e+02$	$5.027833e-06$	2
6	$1.767212e+02$	$4.540506e-13$	1

Table 2: Convergence of SQP-method for (PT^{VC})

We observe that the convergence behavior of the SQP-method applied to (PT^{VC}) and (PT^{MY}) is very similar. In the case of the virtual control approach the algorithm needs one SQP-step more than for Moreau-Yosida regularization. The stopping criterion contains an additional term (measuring the error in the virtual control) in the virtual control approach. Therefore, the corresponding criterion is stronger than that one for the Moreau-Yosida regularization. This leads to an additional SQP step.

Concluding, we will compare both of the concepts as the related regularization parameter tends to infinity and zero, respectively. We mention Hintermüller and Kunisch in [14, 13], where path-following methods associated to the Moreau-Yosida regularization parameter are developed. In this numerical test, we will use only a simple nested approach for both of the regularization concepts: the numerical solution of the problem (PT^{VC}) or (PT^{MY}), respectively, is taken as the starting point for the SQP-method with respect to the next regularization parameter. The convergence behavior of the Moreau-Yosida approximation (PT^{MY}) for increasing regularization parameters γ is displayed in Table 3. As expected, the errors $\|\bar{u}_\gamma - \bar{u}\|$ and $\|\bar{y}_\gamma - \bar{y}\|$ are decreasing for increasing parameters γ. Moreover, an influence of the discretization error is visible in the difference of the controls.

γ	$\|\bar{u}_\gamma - \bar{u}\|$	$\|\bar{y}_\gamma - \bar{y}\|$	$\#it_{SQP}$	$\#it_{AS}$
20	$5.114051e-01$	$1.495545e-02$	8	34
40	$3.161757e-01$	$7.893723e-03$	3	5
80	$1.853852e-01$	$4.056569e-03$	2	3
160	$1.059317e-01$	$2.053420e-03$	2	3
320	$6.072430e-02$	$1.030834e-03$	2	3
640	$3.559594e-02$	$5.152590e-04$	2	2
1280	$2.176439e-02$	$2.568771e-04$	2	3
2560	$1.442802e-02$	$1.277243e-04$	2	2
5120	$1.093492e-02$	$6.323253e-05$	2	2
10240	$9.458933e-03$	$3.104869e-05$	2	2
20480	$8.883362e-03$	$1.500880e-05$	2	2
40960	$8.656709e-03$	$7.063147e-06$	2	3

Table 3: Convergence of (PT^{MY})

We proceed with an analogous observation for the virtual control concept (PT^{VC}) by setting $\varepsilon = 1/\gamma$ and the parameter functions

$$\psi(\varepsilon) \equiv 1, \quad \phi(\varepsilon) = \sqrt{\varepsilon}, \quad \xi(\varepsilon) = \sqrt{\varepsilon}.$$

The results are shown in Table 4. The regularization error in the state and the control is decreasing as the regularization parameter ε tends to zero. Comparing both regularization concepts, there is no significant difference in the development of the errors and the iteration numbers. The slightly higher numbers of iterations for the Primal dual active set strategy were explained by the different quantities δ_γ and δ_ε for the stopping criteria of the SQP-algorithm.

ε	$\\|\bar{u}_\varepsilon - \bar{u}\\|$	$\\|\bar{y}_\varepsilon - \bar{y}\\|$	$\#it_{SQP}$	$\#it_{AS}$
$5.e-02$	$7.529252e-01$	$3.348741e-02$	8	36
$2.5e-02$	$4.016954e-01$	$1.860292e-02$	3	5
$1.25e-02$	$2.130163e-01$	$9.828646e-03$	3	5
$6.25e-03$	$1.157951e-01$	$5.045643e-03$	3	4
$3.125e-03$	$6.472462e-02$	$2.555012e-03$	3	4
$1.5625e-03$	$3.714016e-02$	$1.284863e-03$	3	4
$7.8125e-04$	$2.233563e-02$	$6.435689e-04$	2	2
$3.90625e-04$	$1.470637e-02$	$3.215488e-04$	2	2
$1.953125e-04$	$1.109796e-02$	$1.603108e-04$	2	2
$9.765625e-05$	$9.563392e-03$	$7.970102e-05$	2	3
$4.882813e-05$	$8.951080e-03$	$3.944239e-05$	2	3
$2.441406e-05$	$8.700583e-03$	$1.936622e-05$	2	4

Table 4: Convergence of (PT^{VC})

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