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All functions are locally s-harmonic up to a small error

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ABSTRACT. We show that we can approximate every function $f \in C^k(\overline{B_1})$ with a s-harmonic function in B_1 that vanishes outside a compact set.

That is, s-harmonic functions are dense in $C^k_{\rm loc}$. This result is clearly in contrast with the rigidity of harmonic functions in the classical case and can be viewed as a purely nonlocal feature.

1. INTRODUCTION

It is a well-known fact that harmonic functions are very rigid. For instance, in dimension 1, they reduce to a linear function and, in any dimension, they never possess local extrema.

The goal of this paper is to show that the situation for fractional harmonic functions is completely different, namely one can fix any function in a given domain and find a s-harmonic function arbitrarily close to it.

Heuristically speaking, the reason for this phenomenon is that while classical harmonic functions are determined once their trace on the boundary is fixed, in the fractional setting the operator sees all the data outside the domain. Hence, a careful choice of these data allows a s-harmonic function to “bend up and down” basically without any restriction.

The rigorous statement of this fact is in the following Theorem 1.1. For this, we recall that, given $s \in (0,1)$, the fractional Laplace operator of a function u is defined (up to a normalizing constant) as

$$(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy.$$

We refer to [4, 6, 8, 9] for other equivalent definitions, motivations and applications.

Theorem 1.1. *Fix $k \in \mathbb{N}$. Then, given any function $f \in C^k(\overline{B_1})$ and any $\epsilon > 0$, there exist $R > 1$ and $u \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ such that*

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1, \\ u = 0 & \text{in } \mathbb{R}^n \setminus B_R \end{cases}$$

and

$$\|f - u\|_{C^k(B_1)} \leqslant \epsilon.$$

As usual, in Theorem 1.1, we have denoted by $C^k(\overline{B_1})$ the space of all the functions $f : \overline{B_1} \to \mathbb{R}$ that possess an extension $\tilde f \in C^k(B_{1+\mu})$ (i.e. $\tilde f = f$ in B_1), for some $\mu > 0$.

We also mention that an important rigidity feature for classical harmonic functions is imposed by Harnack inequality: namely if u is harmonic and non-negative in B_1 then $u(x)$ and $u(y)$ are comparable for any $x, y \in B_{1/2}$. A striking difference with the nonlocal case is that this type of Harnack inequality fails for the fractional Laplacian (namely it is necessary to require that u is non-negative in the whole of $\mathbb{R}^n$ and not only in B_1, see e.g. Theorem 2.2 in [5]). As an application of Theorem 1.1, we point out that one can construct examples of s-harmonic functions with a “wild” behavior, that oscillate as many times as we want, and reach interior extrema basically at any prescribed point. In particular, one can construct s-harmonic functions to be used as barriers basically without any geometric restriction.

As a final observation, we would like to stress that, while Theorem 1.1 reveals a purely nonlocal phenomenon, a similar result does not hold for any nonlocal operator. For instance, it is not possible to replace “s-harmonic functions” with “nonlocal minimal surfaces” in the statement of Theorem 1.1, that is it is not true that any graph may be locally approximated by nonlocal minimal surfaces. Indeed, the uniform density estimates satisfied by the nonlocal minimal surfaces prescribe a severe geometric restriction that prevent the formation of sharp edges and thin spikes.

We refer to [2] for the definition of nonlocal minimal surfaces and for their density properties: as a matter of fact, one of the consequence of Theorem 1.1 is that density properties do not hold true for s-harmonic functions, so s-harmonic functions and nonlocal minimal surfaces may have very different behaviors.

The rest of the paper is organized as follows: in Section 2 we collect some preliminary results, such as a (probably well-known) generalization of the Stone-Weierstrass Theorem and the construction of a s-harmonic function in B_1 that has a well-defined growth from the boundary. Then, in Section 3, we construct a s-harmonic function with an arbitrarily large number of derivatives prescribed. This is, in a sense, already the core of our argument, since these types of properties are typical for the fractional case and do not hold for classical harmonic functions. Also, from this result, the proof of Theorem 1.1 will follow via a scaling and approximation method.

2. Preliminary observations

In this section we collect some auxiliary results that will be needed in the rest of the paper.

First of all, we recall a version of the Stone-Weierstrass Theorem for smooth functions. We give a quick proof of it since in general this result is presented only in the continuous setting.

Lemma 2.1. *For any $f \in C^k(\overline{B_1})$ and any $\epsilon > 0$ there exists a polynomial P such that $\|f - P\|_{C^k(B_1)} \leqslant \epsilon$.*

Proof. Without loss of generality we may suppose that $f \in C^k_0(B_2)$. Also, given $\epsilon > 0$ as in the statement of Lemma 2.1, we fix $R > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_R} e^{-|x|^2}\, dx \leqslant \epsilon. \tag{1}$$

Then, we fix $\eta > 0$, to be taken arbitrarily small (possibly in dependence of ϵ and R, which are fixed once and for all), and we take $J_\eta \in \mathbb{N}$ large enough such that

$$\sum_{j > J_\eta} \frac{(-1)^j}{j!\, \eta^j} \leqslant e^{-1/\eta}. \tag{2}$$

Let also

$$\begin{aligned} Q(x) &:= (\pi\eta)^{-n/2} \sum_{j=0}^{J_\eta} \frac{(-1)^j\, |x|^{2j}}{j!\, \eta^j}, \\ P(x) &:= \int_{\mathbb{R}^n} f(y)\, Q(x-y)\, dy, \\ \text{and} \qquad G(x) &:= (\pi\eta)^{-n/2} e^{-|x|^2/\eta}. \end{aligned}$$

We remark that Q is a polynomial in x, hence so is P. Moreover, by a Taylor expansion,

$$G(x) = Q(x) + (\pi\eta)^{-n/2} \sum_{j > J_\eta} \frac{(-1)^j\, |x|^{2j}}{j!\, \eta^j}$$

and so, using (2), we conclude that, for any $x \in B_3$,

$$|G(x) - Q(x)| \leqslant e^{-1/\sqrt{\eta}}, \tag{3}$$

provided that η is sufficiently small.

Now we recall (1) and we observe that, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant k$ and any $x \in B_1$,

$$\begin{aligned} & |D^\alpha (G * f)(x) - D^\alpha f(x)| \\ = & \left| \int_{\mathbb{R}^n} G(y) \Big(D^\alpha f(x-y) - D^\alpha f(x) \Big) \, dy \right| \\ \leqslant & \pi^{-n/2} \int_{\mathbb{R}^n} e^{-|z|^2} \Big| D^\alpha f(x - \sqrt{\eta}\, z) - D^\alpha f(x) \Big| \, dz \\ \leqslant & 2\pi^{-n/2} \epsilon \, \|f\|_{C^k(\mathbb{R}^n)} + \pi^{-n/2} \int_{B_R} e^{-|z|^2} \Big| D^\alpha f(x - \sqrt{\eta}\, z) - D^\alpha f(x) \Big| \, dz \\ \leqslant & C \left(\epsilon + R^n \sup_{z \in B_R} \Big| D^\alpha f(x - \sqrt{\eta}\, z) - D^\alpha f(x) \Big| \right), \end{aligned}$$

for some $C > 0$. Now, if η is sufficiently small, we have that

$$\sup_{|x-y| \leqslant \sqrt{\eta}\, R} \Big| D^\alpha f(x) - D^\alpha f(y) \Big| \leqslant R^{-n} \epsilon,$$

thus we conclude that

$$|D^\alpha (G * f)(x) - D^\alpha f(x)| \leqslant C\epsilon, \tag{4}$$

for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant k$ and any $x \in B_1$, for a suitable $C > 0$.

Furthermore, using (3) we see that, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant k$ and any $x \in B_1$,

$$\begin{aligned} |D^\alpha (G * f)(x) - D^\alpha P(x)| &= |D^\alpha (G * f)(x) - D^\alpha (Q * f)(x)| \\ &= \left| \int_{B_3} \Big(G(y) - Q(y) \Big) \, D^\alpha f(x-y) \, dy \right| \\ &\leqslant C \, \|f\|_{C^k(\mathbb{R}^n)} \, e^{-1/\sqrt{\eta}} \\ &\leqslant \epsilon, \end{aligned}$$

as long as η is small enough. From this and (4) we obtain

$$\|f - P\|_{C^k(\mathbb{R}^n)} \leqslant \|f - (G * f)\|_{C^k(\mathbb{R}^n)} + \|(G * f) - P\|_{C^k(\mathbb{R}^n)} \leqslant C\epsilon,$$

for some $C > 0$, which is the desired result, up to renaming ϵ. □

Now, we construct a s-harmonic function in B_1 that has a well-defined growth from the boundary:

Lemma 2.2. *Let $\bar\psi \in C^\infty(\mathbb{R}, [0,1])$ such that $\bar\psi(t) = 0$ for any $t \in \mathbb{R} \setminus (2,3)$ and $\bar\psi(t) > 0$ for any $t \in (2,3)$.*

Let $\psi_0(x) := \bar\psi(|x|)$ and $\psi \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ be the solution of

$$\begin{cases} (-\Delta)^s \psi = 0 & \text{in } B_1, \\ \psi = \psi_0 & \text{in } \mathbb{R}^n \setminus B_1. \end{cases}$$

Then, if $x \in \partial B_{1-\epsilon}$, we have that

$$\psi(x) = \kappa \, \epsilon^s + o(\epsilon^s) \tag{5}$$

as $\epsilon \to 0^+$, for some $\kappa > 0$.

Proof. We notice that the function $\psi \in H^s(\mathbb{R}^n)$ may be constructed by the direct method of the calculus of variations, and also $\psi \in C^s(\mathbb{R}^n)$, see e.g. [7].

Also, we use the Poisson Kernel representation (see e.g. [1,6]) to write, for any $x \in B_1$,

$$\begin{aligned}\psi(x) &= c\int_{\mathbb{R}^n\setminus B_1} \frac{\psi_0(y)\,(1-|x|^2)^s}{(|y|^2-1)^s\,|x-y|^n}\,dy\\ &= c\,(1-|x|^2)^s\int_2^3\left[\int_{S^{n-1}} \frac{\rho^{n-1}\bar\psi(\rho)}{(\rho^2-1)^s\,|x-\rho\omega|^n}\,d\omega\right]\,d\rho,\end{aligned}$$

for some $c > 0$. Now we take $x \in B_1$, with $|x| = 1-\epsilon$, and we obtain

$$\begin{aligned}\psi(x) &= c\,(2\epsilon-\epsilon^2)^s\int_2^3\left[\int_{S^{n-1}} \frac{\rho^{n-1}\bar\psi(\rho)}{(\rho^2-1)^s\,|(1-\epsilon)e_1-\rho\omega|^n}\,d\omega\right]\,d\rho\\ &= 2^s\,c\,\epsilon^s\int_2^3\left[\int_{S^{n-1}} \frac{\rho^{n-1}\bar\psi(\rho)}{(\rho^2-1)^s\,|e_1-\rho\omega|^n}\,d\omega\right]\,d\rho + o(\epsilon^s)\\ &= \kappa\,\epsilon^s + o(\epsilon^s),\end{aligned}$$

for some $\kappa > 0$, as desired. □

We observe that alternative proofs of Lemma 2.2 may be obtained from a boundary Harnack inequality in the extended problem and from explicit barriers, see [3, 7].

By blowing up the functions constructed in Lemma 2.2 we obtain the existence of a sequence of s-harmonic functions approaching $(x\cdot e)_+^s$, for a fixed unit vector e, as stated below:

Corollary 2.3. *Fixed $e \in \partial B_1$, there exists a sequence of functions $v_{e,j} \in H^s(\mathbb{R}^n)\cap C^s(\mathbb{R}^n)$ such that $(-\Delta)^s v_{e,j} = 0$ in $B_1(e)$, $v_{e,j} = 0$ in $\mathbb{R}^n \setminus B_{4j}(e)$, and*

$$v_{e,j}(x) \to \kappa(x\cdot e)_+^s \ \text{ in } L^1(B_1(e)),$$

as $j \to +\infty$, for some $\kappa > 0$.

Proof. Let ψ be as in Lemma 2.2 and

$$v_{e,j}(x) := j^s\psi(j^{-1}x - e).$$

The s-harmonicity of $v_{e,j}$ and the property of its support can be derived from the ones of ψ. We now prove the convergence. For this, given $x \in B_1(e)$ we write $p_j := j^{-1}x - e$ and $\epsilon_j := 1 - |p_j| = 1 - |j^{-1}x - e|$. We remark that

$$1 > |x-e|^2 = |x|^2 - 2x\cdot e + 1,$$

which implies that

$$|x|^2 < 2x\cdot e, \text{ and } x\cdot e > 0 \text{ for all } x \in B_1(e). \tag{6}$$

As a consequence

$$|p_j|^2 = |j^{-1}x-e|^2 = j^{-2}|x|^2 + 1 - 2j^{-1}x\cdot e = 1 - 2j^{-1}(x\cdot e)_+ + o(j^{-1})\,(x\cdot e)_+^2$$

and so

$$\epsilon_j = j^{-1}\,(1+o(1))\,(x\cdot e)_+.$$

Therefore, using (5), we have

$$\begin{aligned}v_{e,j}(x) &= j^s\psi(p_j)\\ &= j^s\big(\kappa\epsilon_j^s + o(\epsilon_j^s)\big)\\ &= j^s\big(\kappa j^{-s}(x\cdot e)_+^s + o(j^{-s})\big)\\ &= \kappa\,(x\cdot e)_+^s + o(1).\end{aligned}$$

Integrating over $B_1(e)$ we obtain the desired convergence. $\square$

3. Spanning the derivative of a function and proof of Theorem 1.1

The main result of this section is that we can find a s-harmonic function with an arbitrarily large number of derivatives prescribed. For this, we use the standard norm notation for a given multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, according to which

$$|\alpha| := \alpha_1 + \dots + \alpha_n.$$

Theorem 3.1. *For any $\beta \in \mathbb{N}^n$ there exist $R > r > 0$, $p \in \mathbb{R}^n$, $v \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ such that*

$$\begin{cases} (-\Delta)^s v = 0 & \text{in } B_r(p), \\ v = 0 & \text{in } \mathbb{R}^n \setminus B_R(p), \end{cases} \tag{7}$$

$$D^\alpha v(p) = 0 \text{ for any } \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leqslant |\beta| - 1, \tag{8}$$

$$D^\alpha v(p) = 0 \text{ for any } \alpha \in \mathbb{N}^n \text{ with } |\alpha| = |\beta| \text{ and } \alpha \neq \beta, \tag{9}$$

$$\text{and } D^\beta v(p) = 1. \tag{10}$$

Proof. We denote by $\mathcal{Z}$ the set containing the couples (v, x) of all functions $v \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ and points $x \in B_r(p)$ that satisfy (7) for some $R > r > 0$ and $p \in \mathbb{R}^n$.

We let

$$N := \sum_{j=0}^{|\beta|} n^j.$$

To any $(v, x) \in \mathcal{Z}$ we can associate a vector in $\mathbb{R}^N$ by listing all the derivatives of v up to order $|\beta|$ evaluated at x, that is

$$\Big(D^\alpha v(x) \Big)_{|\alpha| \leqslant |\beta|} \in \mathbb{R}^N.$$

We claim that the vector space spanned by this construction exhausts $\mathbb{R}^N$ (if we prove this, then we obtain (8)–(10) by writing the vector with entry 1 when $\alpha = \beta$ and 0 otherwise as linear combination of the above functions).

Thus we argue by contradiction, assuming that the vector space above does not exhaust $\mathbb{R}^N$ but lies in a subspace. That is, there exists $c = (c_\alpha)_{|\alpha| \leqslant |\beta|} \in \mathbb{R}^N \setminus \{0\}$ such that

$$\sum_{|\alpha| \leqslant |\beta|} c_\alpha D^\alpha v(x) = 0 \tag{11}$$

for any $(v, x) \in \mathcal{Z}$. In particular, fixed any $\xi \in \mathbb{R}^n \setminus \{0\}$ and letting $e := \xi / |\xi|$, we have that (11) holds true when $v := v_{e,j}$ and $x \in B_1(e)$, as warranted by Corollary 2.3.

Accordingly, for every $\varphi \in C_0^\infty(B_1(e))$,

$$\begin{aligned} 0 &= \lim_{j\to+\infty} \int_{\mathbb{R}^n} \sum_{|\alpha|\leqslant|\beta|} c_\alpha D^\alpha v_{e,j}(x)\,\varphi(x)\,dx \\ &= \lim_{j\to+\infty} \int_{\mathbb{R}^n} \sum_{|\alpha|\leqslant|\beta|} (-1)^{|\alpha|} c_\alpha\, v_{e,j}(x)\, D^\alpha\varphi(x)\,dx \\ &= \kappa \int_{\mathbb{R}^n} \sum_{|\alpha|\leqslant|\beta|} (-1)^{|\alpha|} c_\alpha\,(x\cdot e)_+^s\, D^\alpha\varphi(x)\,dx \\ &= \kappa \int_{\mathbb{R}^n} \sum_{|\alpha|\leqslant|\beta|} c_\alpha\, D^\alpha(x\cdot e)_+^s\,\varphi(x)\,dx. \end{aligned}$$

Consequently, for any $x \in B_1(e)$,

$$\sum_{|\alpha|\leqslant|\beta|} c_\alpha\, D^\alpha(x\cdot e)_+^s = 0. \tag{12}$$

Recalling (6), we observe that, for any $x \in B_1(e)$,

$$D^\alpha(x\cdot e)_+^s = s\,(s-1)\,\ldots\,(s-|\alpha|+1)\,(x\cdot e)_+^{s-|\alpha|}\, e_1^{\alpha_1}\,\ldots\, e_n^{\alpha_n}.$$

So we write (12) as

$$\sum_{|\alpha|\leqslant|\beta|} \tilde c_\alpha(x)\,\xi^\alpha = 0, \tag{13}$$

for any $x \in B_1(e)$ and any $\xi \in \mathbb{R}^n \setminus \{0\}$ (and hence for any $\xi \in \mathbb{R}^n$ by continuity), where $\tilde c_\alpha(x) := s\,(s-1)\,\ldots\,(s-|\alpha|+1)\,(x\cdot e)_+^{s-|\alpha|}\, c_\alpha$.

We remark that, for a fixed $x \in B_1(e)$, equation (13) says that a polynomial in the variable ξ is identically equal to 0. Therefore all its coefficients must vanish, namely

$$s\,(s-1)\,\ldots\,(s-|\alpha|+1)\,(x\cdot e)_+^{s-|\alpha|}\, c_\alpha = 0 \tag{14}$$

for any $x \in B_1(e)$ and any $|\alpha| \leqslant |\beta|$.

Notice that none of the terms s, $(s-1)$, $\ldots$, $(s-|\alpha|+1)$ vanish since s is not an integer. Using this and (6), we deduce from (14) that $c_\alpha = 0$ for any $|\alpha| \leqslant |\beta|$, that is $c = 0$, against our assumptions. □

We stress that Theorem 3.1 reflects a purely nonlocal feature. Indeed, in the local case (i.e. when $s = 1$) the statement of Theorem 3.1 would be clearly false when $|m| \geqslant 2$, since the sum of the pure second derivatives of any harmonic function must vanish and cannot sum up to 1.

With the aid of Theorem 3.1, we can now complete the proof of Theorem 1.1:

Proof of Theorem 1.1. By Lemma 2.1, we can reduce ourselves to the case in which f is a polynomial. Consequently, the linearity of the fractional Laplace operator allows us to reduce to the case in which f is a monomial, say

$$f(x) = \frac{x^\beta}{\beta!}$$

for some $\beta \in \mathbb{N}^n$. Then we take v as in Theorem 3.1 and we define

$$u(x) := \eta^{-|\beta|} v(\eta x + p),$$

with $\eta \in (0, 1/2)$ to be taken conveniently small in the sequel (in dependence of ϵ that is fixed in the statement of Theorem 1.1).

Let also $g(x) := u(x) - f(x) = u(x) - (\beta!)^{-1} x^\beta$. By Theorem 3.1 we know that $D^\alpha g(0) = 0$ for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant |\beta|$. Furthermore, if $|\alpha| \geqslant |\beta| + 1$,

$$|D^\alpha g(x)| = \eta^{|\alpha|-|\beta|} |D^\alpha v(\eta x + p)| \leqslant C_{|\alpha|}\, \eta \|v\|_{C^{|\alpha|}(B_{1/2}(p))},$$

for any $x \in B_1$, for some $C_{|\alpha|} > 0$. As a consequence, defining $k' := k + |\beta| + 1$ and fixed any $\gamma \in \mathbb{N}^n$ with $|\gamma| \leqslant k' - 1$ and any $x \in B_1$, we obtain by a Taylor expansion that

$$D^\gamma g(x) = \sum_{|\beta|+1 \leqslant |\gamma|+|\alpha| \leqslant k'-1} \frac{D^{\gamma+\alpha} g(0)}{\alpha!} x^\alpha + \sum_{|\gamma|+|\alpha|=k'} \frac{k'}{\alpha!} \int_0^1 (1-t)^{k'-1} D^{\gamma+\alpha} g(tx)\, dt\, x^\alpha$$

and so $|D^\gamma g(x)| \leqslant C\eta$, with $C > 0$ possibly depending also on v.

Since this is valid for any $x \in B_1$ we obtain that

$$\|u - f\|_{C^k(B_1)} = \|g\|_{C^k(B_1)} \leqslant \|g\|_{C^{k'-1}(B_1)} \leqslant C\eta,$$

for some $C > 0$, which implies the statement of Theorem 1.1 as long as $\eta \in (0, C^{-1}\epsilon)$. □

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