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Recent trends and views on elliptic quasi-variational inequalities

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Abstract

We consider state-of-the-art methods, theoretical limitations, and open problems in elliptic Quasi-Variational Inequalities (QVIs). This involves the development of solution algorithms in function space, existence theory, and the study of optimization problems with QVI constraints. We address the range of applicability and theoretical limitations of fixed point and other popular solution algorithms, also based on the nature of the constraint, e.g., obstacle and gradient-type. For optimization problems with QVI constraints, we study novel formulations that capture the multivalued nature of the solution mapping to the QVI, and generalized differentiability concepts appropriate for such problems.

1 Introduction

Quasi-Variational Inequalities represent a specific subclass of quasi-equilibrium problems in which non-convexity and non-smoothness are present. They play an important role in the modelling of complex phenomena in applied sciences, engineering, and economy, where compliancy or other state dependent bound constraints have to be taken care of. The nonlinear nature of the constraint set challenges the derivation of existence results and the design and analysis of associated solution algorithms.

In the majority of available models, the state dependent constraint is of the form

$$\psi(Gy) \leq \Phi(y),$$

where ψ is a real-valued nonlinear function, G a linear operator, and Φ a nonlinear operator that is of superposition type or it is defined by the solution mapping of a nonlinear partial differential equation (PDE). For example, in the case of unilateral constraints, $\psi(x) = x$ and $G = \mathrm{id}$, and for gradient constraints, $\psi(x) = |x|$ and $G = \nabla$ is the weak gradient. Applications involving these restrictions include, but are not limited to, the magnetization of superconductors, Maxwell systems, thermohydraulics, image processing, game theory, surface growth of granular (cohensionless) materials, hydrology, and solid and continuum mechanics. For more details, we refer the reader to [21, 25, 34, 52, 54, 56, 64, 65] and the monographs [14, 53].

The goal of this paper is to present state-of-the-art results including mathematical limitations and open questions that arise in the treatment of QVIs. Specific focus topics involve existence of solutions, development of appropriate solvers together with some problematic issues found in the literature, optimal control, and directional differentiability of the QVI solution map.

Due to our aim of keeping the paper compact, we have not been able to include certain important approaches. In particular, in the case of gradient constraints, the QVI can be rewritten as a generalized equation. It then follows that these QVIs become a particular instance of a more general problem class; see, e.g., [48, 51]. The latter approach was pioneered by Kenmochi and collaborators, and further work

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can be found in [24, 26, 46, 47]. Also, we have not included the L^∞ contraction results from Hanouzet and Joly which are well documented in [31, 32] and [15]. As we focus on the infinite dimensional setting in this paper, we have not included recent finite dimensional solvers associated with KKT-type and augmented Lagrangian methods; see [22, 23, 35, 49, 50]. In a similar vein, we have avoided discretization issues of closed convex sets which are required for consistency of numerical schemes and are deeply related to the density of smooth functions on the aforementioned sets; see [41, 43].

The paper is organized as follows. The class of problems under consideration is described in section 1.1, where the basic functional analytic framework is established, and solutions to the QVIs of interest are equivalently described as fixed points of a specific nonlinear map T. In section 2, we consider some existence results involving compactness or increasing properties of the map T. Furthermore, we provide sufficient conditions for both properties and mention open questions concerning both approaches. Section 3 concerns iterative methods for solving QVIs. We state some results for obstacle, gradient, and more general constraints. Also, we focus on an unfortunate trend of the QVI literature that intends to extend the technique of the Lions–Stampacchia existence result to the QVI setting. We show that in general the approach is rather restrictive and that the assumption of the Lipschitz continuity of the projection map $\mathbf{K} \mapsto \mathrm{P}_{\mathbf{K}}$, frequently made, does not hold in general. Subsequently, we consider iterations that converge in case of multiple solutions and regularization approaches of Moreau–Yosida and Gerhardt type. We finalize the section by addressing drawbacks associated with the simple fixed point iteration $y_n = T(y_{n-1})$. In section 4, we state optimal control problems with QVI constraints that take into account the multivalued nature of the solution set. In particular, utilizing a control reduced form of the problem leads to a formulation in terms of minimum and maximum solutions to the QVI. A newly established directional differentiability result for the QVI map is provided in section 5, where the classical result of Mignot is extended accordingly to the QVI framework.

1.1 The basic setting and problem formulation

We consider V to be a reflexive real Banach space of (equivalence) classes of maps of the type $v : \Omega \to \mathbb{R}$ for some Lipschitz domain $\Omega \subset \mathbb{R}^N$ with $N \in \mathbb{N}$ and norm denoted by $\|\cdot\|_V$. Its topological dual is denoted by V' and the pairing between V' and V is given by $\langle \cdot, \cdot \rangle$. If V is a Hilbert space, then $(\cdot, \cdot)$ denotes its inner product. For a sequence $\{v_n\}$ in V, strong and weak convergence to $v \in V$ are written as "$v_n \to v$" and "$v_n \rightharpoonup v$", respectively.

For a map $K\colon V \to W$, where W is a Banach space, we say that K is *completely continuous* if $v_n \rightharpoonup v$ in V implies $K(v_n) \to K(v)$ in W. Since V is reflexive, a completely continuous map is compact; see [71, Chapter II, Lemma 1.1].

Throughout the paper we consider a (possibly nonlinear) operator $A\colon V \to V'$ that is Lipschitz continuous and uniformly monotone, i.e., there exist constants $c > 0$ and $C > 0$ such that for all $u, v \in V$,

$$\|A(u) - A(v)\|_{V'} \le C\|u - v\|_V, \tag{A1}$$

and

$$\langle A(u) - A(v), u - v\rangle \ge c\|u - v\|_V^r, \tag{A2}$$

for some constant $r > 1$. If V is a Hilbert space, then $r = 2$. In addition, we assume that $A(0) = 0$.

The typical setting that we consider here is with $V := W_0^{1,p}(\Omega)$, with $\Omega \subset \mathbb{R}^N$ a bounded Lipschitz domain, $2 \le p < +\infty$, and $A := -\Delta_p$, the $p-$Laplacian, given by

$$\langle -\Delta_p(u), v\rangle := \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x, \quad \text{for } u, v \in W_0^{1,p}(\Omega).$$

In this case, $c = 1$ and $r = p$.

The general problem class under consideration is given as follows.

Problem $(\mathrm{P_{QVI}})$: Given $f \in V'$,

$$\text{find } y \in \mathbf{K}(y) : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(y). \tag{$\mathrm{P_{QVI}}$}$$

The general structure of $v \mapsto \mathbf{K}(v)$ is given by

$$\mathbf{K}(v) := \{w \in V : \psi(Gw) \leq \Phi(v)\}, \tag{1}$$

where $\Phi(v)\colon \Omega \to \mathbb{R}$ is a measurable function for each v and "$v \leq w$" means that $v(x) \leq w(x)$ for almost all (f.a.a.) $x \in \Omega$, unless stated otherwise. We assume that $G \in \mathscr{L}(V, L^p(\Omega)^d)$ for some $1 < p < +\infty$ and $d \in \mathbb{N}$, that is, $G\colon V \to L^p(\Omega)^d$ is linear and bounded. Additionally, we suppose that $\psi\colon \mathbb{R}^d \to \mathbb{R}$ is convex. For the sake of simplicity, we assume that $\mathbf{K}(v)$ is always non-empty for each v. The closedness and convexity of $\mathbf{K}(v)$ follow from the assumptions invoked here. Additionally, we assume that $v \mapsto \max(0, v)$, and $v \mapsto \min(0, v)$ are continuous with respect to the weak and strong topologies of V.

We distinguish at least two notable cases both for $V = W_0^{1,p}(\Omega)$. If $\psi(Gw) = w$, we refer to the problem as the *obstacle* case. If $\psi(x) = |x|$ and $G = \nabla$ is the weak gradient so that $G\colon W_0^{1,p}(\Omega) \to L^p(\Omega)$, we refer to the problem as the *gradient* case.

We denote the solution set to $(\mathrm{P_{QVI}})$ for a given $f \in V'$ by $\mathbf{Q}(f)$, and note that in general $\mathbf{Q}(f)$ contains more than one element. It is convenient to characterize $\mathbf{Q}(f)$ as the set of fixed points of a certain map. In this light, consider $\mathbf{K} \subset V$ non-empty, closed and convex. Then for any $f \in V'$, we define $S(f, \mathbf{K})$ as the unique solution to:

$$\text{Find } y \in \mathbf{K} : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}. \tag{2}$$

Also, for the map $v \mapsto \mathbf{K}(v)$ given as above, we consider

$$T(v) := S(f, \mathbf{K}(v)). \tag{3}$$

It then follows that solutions to $(\mathrm{P_{QVI}})$ are equivalently defined as solutions to

$$T(v) = v.$$

In general for an operator R, we denote the set of fixed points by $\mathrm{Fix}(R)$.

2 Some existence theory

In this section we provide an overview of techniques available to prove existence of solutions to QVIs and the limitations and caveats associated with the utilized techniques. We focus on compactness results and ordering approaches. Contraction methods, however, are left for the section on solution algorithms.

2.1 Compactness and Mosco convergence

One approach to prove existence of a fixed point of T is based on compactness of the map T. In particular, since V is reflexive, it is enough to consider the complete continuity of T, i.e., given $v_n \rightharpoonup v$, then $T(v_n) \to T(v)$ in V; see [71, Chapter II, Lemma 1.1.]. Then, a suitable fixed point theorem yields existence. Note, however, that this is directly associated with a notion of set convergence for $\{\mathbf{K}(v_n)\}$, as introduced by Mosco; see [62].

Definition 1 (Mosco convergence) *Let $\mathbf{K}$ and $\mathbf{K}_n$, for each $n \in \mathbb{N}$, be non-empty, closed and convex subsets of V. Then the sequence $\{\mathbf{K}_n\}$ is said to converge to $\mathbf{K}$ in the sense of Mosco as $n \to \infty$, denoted by $\mathbf{K}_n \xrightarrow{\mathrm{M}} \mathbf{K}$, if the following two conditions are fulfilled:*

(i) *For each $w \in \mathbf{K}$, there exists $\{w_{n'}\}$ such that $w_{n'} \in \mathbf{K}_{n'}$ for $n' \in \mathbb{N}' \subset \mathbb{N}$ and $w_{n'} \to w$ in V.*

(ii) *If $w_n \in \mathbf{K}_n$ and $w_n \rightharpoonup w$ in V along a subsequence, then $w \in \mathbf{K}$.*

The importance of Mosco convergence lies in the following continuity result: Let $f_n \to f$ in V', then

$$\mathbf{K}_n \xrightarrow{\mathrm{M}} \mathbf{K} \quad \text{implies} \quad S(f_n, \mathbf{K}_n) \to S(f, \mathbf{K}) \text{ in } V.$$

The proof can be found in [66]. The above fact implies that if $v_n \rightharpoonup v$ in V yields $\mathbf{K}(v_n) \xrightarrow{\mathrm{M}} \mathbf{K}(v)$, then $T\colon V \to V$ is compact. Using $v = 0$ in (2), we observe that $T(V) \subset \overline{B}_{c^{-1}\|f\|_{V'}}(0;V)$, the closed ball in V of radius $c^{-1}\|f\|_{V'}$ and center at 0. Hence by Schauder's fixed point theorem, the equation $T(y) = y$ has solutions in V.

The full characterization of Mosco convergence of $\{\mathbf{K}(v_n)\}$ based on properties of Φ, ψ, and G, is a complex task and to this day, only partial answers are available. Specifically, condition (i) in Definition 1, commonly referred to as the *recovery sequence condition*, is delicate to check in applications, while (ii) admits the following simple and general characterization.

Proposition 2.1 *Suppose that $\Phi\colon V \to L^q(\Omega)$, for some $1 \le q \le +\infty$, is completely continuous, and $v_n \rightharpoonup v$ in V. Then (ii) in Definition 1 holds true for $\mathbf{K}_n = \mathbf{K}(v_n)$ and $\mathbf{K} = \mathbf{K}(v)$.*

Proof. For $w_n \in \mathbf{K}(v_n)$, we have $\psi(Gw_n) \le \Phi(v_n)$, and if $w_n \rightharpoonup w$ in V, it follows that $Gw_n \rightharpoonup Gw$ in $L^p(\Omega)^d$. By Mazur's lemma, there exists $z_n = \sum_{k=n}^{N(n)} \alpha(n)_k G w_k$ where $\sum_{k=n}^{N(n)} \alpha(n)_k = 1$ and $\alpha(n)_k \ge 0$ such that $z_n \to Gw$ in $L^p(\Omega)^d$. Since $\psi\colon \mathbb{R}^d \to \mathbb{R}$ is convex,

$$\psi(z_n) \le \sum_{k=n}^{N(n)} \alpha(n)_k \psi(Gw_k) \le \Phi(v_n).$$

As $v_n \rightharpoonup v$ in V, we have $\Phi(v_n) \to \Phi(v)$ in $L^q(\Omega)$. Hence, we obtain $w \in \mathbf{K}(v)$ by taking the limit above (over some subsequence converging in the pointwise almost everywhere sense). ■

Perhaps the simplest situation in which (i) holds is the obstacle case with $\Phi\colon V \to V$ completely continuous. Let $w \le \Phi(v)$ be arbitrary and $v_n \rightharpoonup v$ in V, and define $w_n := \min(w, \Phi(v_n))$ so that $w_n \le \Phi(w_n)$. Since $\Phi(v_n) \to \Phi(v)$ in V, it follows that $w_n \to w$ in V. Consequently (i) holds true. Note that we assume that $V \ni z \mapsto \min(0,z) \in V$ is continuous. The relaxation of the complete continuity assumption for Φ is an arduous task that we consider in what follows.

2.1.1 The result of Boccardo and Murat

A typical function space setting for our focus problem $(\mathrm{P_{QVI}})$ is given by $V = W_0^{1,p}(\Omega)$, for $1 < p < +\infty$, and obstacle-type constraints. As seen above, if $\Phi\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is completely continuous, then the map T is compact. This can be relaxed substantially by means of the compactness result of Murat in [63]. It states that if $F_n \rightharpoonup F$ in $H^{-1}(\Omega)$ with $F_n \geq 0$ for all $n \in \mathbb{N}$, then $F_n \to F$ in $W^{-1,q}(\Omega)$ with $q < 2$. Here, $F_n \geq 0$ refers to $\langle F_n, \sigma \rangle \geq 0$ for all $\sigma \in V$ with $\sigma \geq 0$. Moreover, the regularity of $\partial\Omega$ can be dropped and the result still remains intact [19]. In our setting, this result leads to the following useful assertion; see [17, 18].

Theorem 2.1 (Boccardo-Murat) *Suppose that $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ implies $\Phi(v_n) \rightharpoonup \Phi(v)$ in $W^{1,q}(\Omega)$ or $W_0^{1,q}(\Omega)$ for some $q > p$. Then $\mathbf{K}(v_n) \xrightarrow{\mathrm{M}} \mathbf{K}(v)$.*

We note that counterexamples can be constructed for $q = p$. In words, the above result relies on the fact that Φ realizes an increase in regularity and preserves weak continuity.

Open problems. *For QVIs with similar constraint types as considered here but with fractional order operators A, a result analogous to the one in Theorem 2.1 appears unavailable. For this kind of operators, the QVI can be equivalently formulated in weighted Sobolev spaces, see [4]. In this context, it is an open question whether it is possible to extend the above result of Boccardo and Murat to weighted Sobolev spaces $W_0^{1,p}(\Omega; w)$ for some w in a Muckenhoupt class.*

2.1.2 Gradient and further cases

The cases other than the obstacle one are significantly more difficult, mainly due to the possible nonlinearity $\psi\colon \mathbb{R}^d \to \mathbb{R}$. Here, we consider the setting where $V = W_0^{1,p}(\Omega)$ with $1 < p < +\infty$. The following result is based on [12, 40, 54]

Proposition 2.2 *Let $G \in \mathscr{L}(W_0^{1,p}(\Omega), L^p(\Omega)^d)$ for some $d \in \mathbb{N}$, and let $\psi\colon \mathbb{R}^d \to \mathbb{R}$ be (positive) homogeneous of degree one, i.e., $\psi(tx) = t\psi(x)$ for any $x \in \mathbb{R}^d$ and $t > 0$. Suppose that $\Phi\colon W_0^{1,p}(\Omega) \to L^\infty_\eta(\Omega) \subset L^\infty(\Omega)$ is completely continuous, where $L^\infty_\eta(\Omega) := \{v \in L^\infty(\Omega) : v \geq \eta > 0 \text{ a.e.}\}$. Then, we have that*

$$v_n \rightharpoonup v \text{ in } W_0^{1,p}(\Omega) \quad \text{implies} \quad \mathbf{K}(v_n) \xrightarrow{\mathrm{M}} \mathbf{K}(v).$$

Proof. First note that by assumption, $\Phi\colon W_0^{1,p}(\Omega) \to L^p(\Omega)$ is also completely continuous. Thus, by Proposition 2.1 we only need to prove the recovery sequence part for Mosco convergence. For this purpose and for $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$, define

$$\beta_n := \left(1 + \frac{\|\Phi(v_n) - \Phi(v)\|_{L^\infty}}{\eta}\right)^{-1}.$$

If $\psi(Gw) \leq \Phi(v)$, then it follows for $w_n := \beta_n w$ that $w_n \to w$ in $W_0^{1,p}(\Omega)$ and $\psi(Gw_n) \leq \Phi(v_n)$ (see [40]) which finishes the proof. ■

We note that the previous result only provides *sufficient* conditions for Mosco convergence; this leads to another open problem.

Open problems. *Find sufficient and necessary conditions on ϕ_n, ϕ such that*

$$\{w \in W_0^{1,p}(\Omega) : \psi(Gw) \leq \phi_n\} \xrightarrow{\mathrm{M}} \{w \in W_0^{1,p}(\Omega) : \psi(Gw) \leq \phi\}.$$

Similarly, it is an open question whether $\phi_n \rightharpoonup \phi$ in $W^{1,q}(\Omega)$ for some q suffices to guarantee the above Mosco convergence in the gradient case by other means than embeddings.

2.2 Order approaches

We consider now an approach based on order that was pioneered by Tartar; see [72] and also [8, Chapter 15, §15.2]. Let (V,H,V') be a Gelfand triple of Hilbert spaces, that is, we have $V \hookrightarrow H \hookrightarrow V'$, where the embedding $V \hookrightarrow H$ is dense and continuous, and H is identified with its topological dual H' so that the embedding $H \hookrightarrow V'$ is also dense and continuous. Within this section, $(\cdot,\cdot)$ denotes the inner product in H.

We assume that $H_+ \subset H$ is a convex cone with

$$H_+ = \{v \in H : (v,y) \geq 0 \text{ for all } y \in H_+\}.$$

Based on this, we use the following ordering denoted by "$\leq$":

$$x \leq y \quad \text{if and only if} \quad y - x \in H_+.$$

For $x \in H$, we have the decomposition $x = x^+ - x^- \in H_+ - H_+$ with $(x^+, x^-) = 0$ such that x^+ denotes the orthogonal projection onto H_+ and $x^- = x - x^+$ the one onto $H_- = -H_+$. The infimum and supremum of two elements $x, y \in H$ are defined as $\sup(x,y) := x + (y-x)^+$ and $\inf(x,y) := x - (x-y)^+$ respectively.

The supremum of an arbitrary subset of H that is bounded (in the order) above is also correctly defined since H is Dedekind complete: A set $\{x_i\}_{i\in J}$ where J is completely ordered and bounded from above implies that $\{x_i\}_{i\in J}$ is a generalized Cauchy sequence in H (see [9, Chapter 15, §15.2, Proposition 1]). From this Dedekind completeness follows (see [2, Chapter 4, Theorem 4.9 and Corollary 4.10]). This additionally implies that norm convergence preserves order. Indeed, if $z_n \leq y_n$ for each $n \in \mathbb{N}$ and $z_n \to z$ and $y_n \to y$ both in H, then $z \leq y$.

Finally, we assume that

$$y \in V \Rightarrow y^+ \in V \qquad \text{and} \qquad \exists \mu > 0 : \|y^+\|_V \leq \mu \|y\|_V, \forall y \in V.$$

Then the order in H induces one in V', as well. In fact, for $f, g \in V'$, we write $f \leq g$ if $\langle f, \phi\rangle \leq \langle g, \phi\rangle$ for all $\phi \in V_+ := V \cap H_+$ and define $V'_+ := \{f \in V' : f \geq 0\}$.

The typical example in this framework is given by the Gelfand triple $(V,H,V') = (H_0^1(\Omega), L^2(\Omega), H^{-1}(\Omega))$. Here, $H_+ = L^2(\Omega)^+$, the set of almost everywhere (a.e.) non-negative functions, and $v \leq w$ denotes that $v(x) \leq w(x)$ for almost all (f.a.a.) $x \in \Omega$.

In this section, we assume that the operator $A \colon V \to V'$ is strictly T-monotone, i.e.,

$$\langle A(y) - A(z), (y-z)^+\rangle > 0, \qquad \forall y, z \in V : (y-z)^+ \neq 0. \tag{A3}$$

In particular, if A is linear, then the above is equivalent to $\langle Ay^-, y^+\rangle \leq 0$ for all $y \in V$, and we have maximum principles available for A. In addition, consider the following definition.

Definition 2 *A map $R\colon V \to V$ is said to be increasing if for $y,z \in V$ we have that*

$$y \le z \quad \textit{implies} \quad R(y) \le R(z).$$

The following general result concerning existence of fixed points for increasing maps is the fundamental tool to prove existence of solutions to problem $(\mathrm{P}_{\mathrm{QVI}})$.

Theorem 2.2 (Tartar–Birkhoff) *Let $R\colon V \to V$ be increasing, and suppose that there exist $\underline{y}, \overline{y} \in V$ such that*

$$\underline{y} \le \overline{y}, \qquad \underline{y} \le R(\underline{y}), \quad \textit{and} \quad R(\overline{y}) \le \overline{y}.$$

Then the set $\mathrm{Fix}(R) \cap [\underline{y}, \overline{y}]$ is non-empty. Furthermore, there exist $y_1, y_2 \in \mathrm{Fix}(R) \cap [\underline{y}, \overline{y}]$ such that

$$y \in \mathrm{Fix}(R) \cap [\underline{y}, \overline{y}] \quad \Rightarrow \quad y \in \mathrm{Fix}(R) \cap [y_1, y_2].$$

The above theorem mainly states that if a map is increasing, has a subsolution y_1 and a supersolution y_2, then it has a fixed point between (with respect to the order induced in H) y_1 and y_2. Moreover, there are minimal and maximal fixed points in $[y_1, y_2]$.

For the map $T\colon V \to V$ to be increasing, some assumptions are required on the structure of $\mathbf{K}$. For this purpose consider the obstacle case and assume that $\Phi\colon V \to H$ is increasing. Also, suppose that $f_{\min} \le f \le f_{\max}$ for some $f_{\min}, f_{\max} \in V'$, and that $\Phi(A^{-1}f_{\min}) \ge A^{-1}f_{\min}$. Then, it follows that

$$\underline{y} = A^{-1}f_{\min} \qquad \text{and} \qquad \overline{y} = A^{-1}f_{\max}$$

are sub- and supersolutions, respectively, of T, and all assumptions of the previous theorem are satisfied. Hence, defining $\mathbf{A}_{\mathrm{ad}} = \{g \in V' : f_{\min} \le g \le f_{\max}\}$, we have the operators

$$\mathsf{m}\colon \mathbf{A}_{\mathrm{ad}} \to V \qquad \text{and} \qquad \mathsf{M}\colon \mathbf{A}_{\mathrm{ad}} \to V$$

that take elements of $\mathbf{A}_{\mathrm{ad}}$ to minimal and maximal solutions to $(\mathrm{P}_{\mathrm{QVI}})$ in the interval $[\underline{y}, \overline{y}] = [A^{-1}f_{\min}, A^{-1}f_{\max}]$.

Open problems. *Characterize the stability of the maps $f \mapsto \mathsf{m}(f)$ and $f \mapsto \mathsf{M}(f)$. Specifically, if $\{f_n\}$ is in $\mathbf{A}_{\mathrm{ad}}$, identify conditions on the sequence $\{f_n\}$ so that*

$$\mathsf{m}(f_n) \to \mathsf{m}(f) \quad \textit{and} \quad \mathsf{M}(f_n) \to \mathsf{M}(f)$$

in H and in V.

3 Solution methods and algorithms

Next we concentrate on solution methods for problem $(\mathrm{P}_{\mathrm{QVI}})$ which are constructive in the sense that they can also be used to show existence of solutions. We focus first on contraction results without the aid of T-monotonicity properties of A, i.e., assumption (A3). In section 3.2, we focus on some problematic tendencies in the literature that attempt to generalize the Lions–Stampacchia existence result on VIs [57] to QVIs. We show that in general, such approaches provide worse results than a simple change of variables and the direct use of (A1). In section 3.3, we exploit ordering properties and consider iterations that converge to $\mathsf{m}(f)$ and $\mathsf{M}(f)$ under appropriate assumptions. Additionally, we consider regularization methods for the constraint $y \in \mathbf{K}(y)$ of the Moreau–Yosida and Gerhardt-type in section 3.4. In the former case, we show how the approach is suitable for Newton-type solvers. We end this section with considerations of the iteration $y_{n+1} = T(y_n)$ when only compactness of T is available.

3.1 Contraction results for T

Uniqueness of solutions to ($\mathrm{P_{QVI}}$) is rarely available. However, in some cases it is possible to obtain that $v \mapsto S(f, \mathbf{K}(v))$ is contractive for a sufficiently small f and with Φ Lipschitz with sufficiently small Lipschitz constant. The interpretation of these prerequisites is as follows: If the Lipschitz constant of Φ satisfies $L_\Phi \ll 1$, then $\Phi(\cdot) \simeq$ constant, and hence it is expected that ($\mathrm{P_{QVI}}$) is close to a variational inequality and admits a unique solution under such assumptions.

3.1.1 Obstacle case

We provide first a simple example associated to the obstacle case that arises when Φ preserves the regularity of the state space (the reason to describe such a simple case is related to the digression in section 3.2).

In the obstacle case, provided that $\Phi\colon V \to V$ is Lipschitz, we can consider the change of variable $z = y - \Phi(v)$. Hence, it is straightforward to prove, via the monotonicity of A, that T satisfies

$$\|T(v_1) - T(v_2)\|_V \leq \frac{1}{c}\|A\Phi(v_1) - A\Phi(v_2)\|_{V'} \leq \frac{C}{c} L_\Phi \|v_1 - v_2\|_V.$$

Consequently, for

$$\frac{C}{c} L_\Phi < 1,$$

the map T has a unique fixed point and the iteration $y_{n+1} = T(y_n)$ converges to this fixed point for any initial $y_0 \in V$. The extent of the usage of this technique is limited to the very case described here. Note also that if $V = H_0^1(\Omega)$, then the assumptions here also imply that $\Phi(v) = 0$ on $\partial\Omega$ in the sense of the trace.

The case $L_\Phi = 1$ may lead to a degenerate situation: Consider $\Phi(y) = y$. Then $y \in \mathbf{K}(y)$ is always satisfied and $v \leq \mathbf{K}(y)$ implies $v - y \leq 0$, so that ($\mathrm{P_{QVI}}$) is equivalent to the problem: Find $y \in V$ such that $Ay \leq f$ in V'. This implies that $A^{-1}g$ is a solution to this problem for every $g \leq f$ in V'.

3.1.2 Gradient and further cases

In other than the obstacle case, contraction results are far more elusive and when available, the contraction rates depend heavily on the regularity and magnitude of the data as we see next. The result is a slight generalization of [40, 42].

We consider the case $V = W_0^{1,p}(\Omega)$ with $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ not necessarily linear, but homogeneous with degree $\beta \geq 1$, i.e., $A(ty) = t^\beta A(y)$ for $t > 0$ and $y \in W_0^{1,p}(\Omega)$, and with monotonicity exponent $r \leq \min(2, p)$ in (A2). We consider $f \in L^{r'}(\Omega) \subset W^{-1,p'}(\Omega)$ where $1/r + 1/r' = 1$ and $1/p + 1/p' = 1$.

Let $G \in \mathscr{L}(W_0^{1,p}(\Omega), L^p(\Omega)^d)$ for some $d \in \mathbb{N}$, and $\psi\colon \mathbb{R}^d \to \mathbb{R}$ such that $\psi(tx) = t\psi(x)$ for $t > 0$. Many examples fit this setting. For instance $G := \nabla$, the weak gradient, or $G := \mathrm{div}$, the weak divergence, together with $\psi(x) = |x|$ corresponding to the Euclidian norm in $\mathbb{R}^N$ or the absolute value respectively. Consider the map $\Phi\colon W_0^{1,p}(\Omega) \to L_\nu^\infty(\Omega)$ defined as $\Phi(u) = \lambda(u)\phi$ where λ is a nonlinear Lipschitz continuous functional and $\phi \in L^\infty(\Omega)$.

Theorem 3.1 ([40]) *In the above described setting, we have*

$$\|T(v_1) - T(v_2)\|_{W_0^{1,p}} \leq L(f)\|v_1 - v_2\|_{W_0^{1,p}},$$

where $L(f) \to 0$ as $\|f\|_{L^{r'}} \to 0$.

Hence, for small data we observe existence of a unique fixed point of T, and thus a unique solution to ($\mathrm{P_{QVI}}$). A proof is given in [40, Theorem B.1].

Relaxing the hypothesis on the structure of Φ typically rules out contraction or even Lipschitz continuity. In order to see this, note that if $\phi_1, \phi_2 \in L^\infty(\Omega)$ and $\mathbf{K}_i := \{v \in W_0^{1,p}(\Omega) : \|\nabla v\|_{\mathbb{R}^N} \leq \phi_i \text{ a.e.}\}$ then

$$\|S(f,\mathbf{K}_1) - S(f,\mathbf{K}_2)\|_{W_0^{1,p}} \leq M(f)\|\phi_1 - \phi_2\|_{L^\infty}^{1/r} \tag{4}$$

where r is the constant in (A2). That is, the map is only Hölder continuous in general; see [40, 70]

Open problems. *The extension of the result of Theorem 3.1 from the rank one case, $\Phi(y) = \lambda(y)\phi$, to the finite rank case, $\Phi(y) = \lambda_1(y)\phi_1 + \lambda_2(y)\phi_2 + \cdots + \lambda_m(y)\phi_m$, is still an open task.*

Additionally, improvements (if possible) on the exponent $1/r$ in (4) have yet to be found, although the Lipschitz continuity result seems unattainable; see also section 3.2.

3.2 The map $\mathbf{K} \mapsto \mathrm{P}_{\mathbf{K}}$ and extensions to Lions–Stampacchia

We restrict ourselves in this section to the Hilbert space setting and describe now a common misleading approach found in the literature. This unfortunate technique is based on aiming to extend the theorem of Lions and Stampacchia in [57] to the QVI framework.

Let $i\colon V \to V'$ denote the duality operator, that is, the canonical isomorphism defined as $\langle iu, v\rangle := (u,v)$, and its inverse $i^{-1} := j$ is the Riesz map for V. Here, problem ($\mathrm{P_{QVI}}$) can be equivalently written as

$$\text{Find } y \in \mathbf{K}(y) : (y - jH_\rho(y), v - y) \geq 0, \quad \forall v \in \mathbf{K}(y)$$

for $H_\rho(w) = iw - \rho(A(w) - f)$ with $w \in V$, and any $\rho > 0$. Then, the existence of a solution to ($\mathrm{P_{QVI}}$) can be transferred to finding $y \in V$ satisfying $y = B_\rho(y)$ with

$$B_\rho(y) := \mathrm{P}_{\mathbf{K}(y)}(y - \rho j(A(y) - f))$$

for some $\rho > 0$. Here $\mathrm{P}_{\mathbf{K}(y)}\colon V \to V \subset \mathbf{K}(y)$ is the projection map, i.e., for any $v \in V$, $\mathrm{P}_{\mathbf{K}(y)}(v)$ is the unique element in $\mathbf{K}(y)$ such that

$$\|\mathrm{P}_{\mathbf{K}(y)}(v) - v\|_V = \inf_{w \in \mathbf{K}(y)} \|w - v\|.$$

In the case where $\Phi(y) = \phi$ for all y, it follows that B_ρ is a contraction provided that $0 < \rho < 2c/C^2$, where c, C are the monotonicity and Lipschitz constant of A, respectively, given in (A1) and (A2). In fact, we have

$$\|B_\rho(v) - B_\rho(w)\|_V \leq \sqrt{1 - 2\rho c + \rho^2 C^2}\,\|v - w\|_V.$$

A significant amount of literature on QVIs is based on trying to extend this result to the quasi-variational setting. This approach relies on the hard assumption

$$\|\mathrm{P}_{\mathbf{K}(y)}(w) - \mathrm{P}_{\mathbf{K}(z)}(w)\|_V \leq \eta\|y - z\|_V \tag{5}$$

for some $0 < \eta < 1$ and all y, z, w in a bounded set in V. This should not be confused with the non-expansiveness of the map $z \mapsto \mathrm{P}_{\mathbf{K}(y)}(z)$, i.e., we have that $\|\mathrm{P}_{\mathbf{K}(y)}(z_1) - \mathrm{P}_{\mathbf{K}(y)}(z_2)\|_V \leq \|z_1 - z_2\|_V$, for all $y, z_1, z_2 \in V$. In general, (5) is *not* valid, and the only framework (in our setting) where it seems to work is in the obstacle type case with $\Phi\colon V \to V$. Indeed, in the latter case we see that the projection map can be rewritten in simpler terms as

$$\mathrm{P}_{\mathbf{K}(y)}(w) = \Phi(y) + \mathrm{P}_{\{z\in V: z\le 0\}}(w - \Phi(y)). \tag{6}$$

Note that it is necessary for this representation that Φ preserves the V regularity. For example if $V = H_0^1(\Omega)$ and Φ maps V into $L^2(\Omega)$ but not into $H_0^1(\Omega)$, this V-regularity requirement is no longer valid.

In case (6) holds, a solution to the QVI is equivalently a fixed point of the map B_ρ now defined as

$$B_\rho(y) := \Phi(y) + \mathrm{P}_{\{z\in V: z\le 0\}}((y - \rho j(A(y) - f) - \Phi(y)),$$

which satisfies

$$\|B_\rho(v) - B_\rho(w)\|_V \le (2L_\Phi + \sqrt{1 - 2\rho c + \rho^2 C^2})\|v - w\|_V.$$

In order for B_ρ to be contractive, a first observation is that we need

$$2L_\Phi + \sqrt{1 - \left(\frac{c}{C}\right)^2} < 1,$$

which implies that

$$\frac{C}{c} L_\Phi < \frac{1}{2}.$$

This is a much more restrictive and convoluted approach than the one described in section 3.1.1, where only $\frac{C}{c}L_\Phi < 1$ is required! Furthermore, the linear convergence rate (in case of a contraction) in this case is worse than the one in section 3.1.1, given by $\frac{C}{c}L_\Phi$.

There is a deep and interesting reason why condition (5) fails in a general setting. The result in question was described by Attouch and Wets in [5–7], and it involves continuity properties of $\mathbf{K} \mapsto \mathrm{P}_{\mathbf{K}}$. This is given in the following section.

3.2.1 The map $\mathbf{K} \mapsto \mathrm{P}_{\mathbf{K}}$

For any closed, non-empty and convex set $\mathbf{K}$ in V, we define the distance function of an element $y \in V$ to the set $\mathbf{K}$ as

$$\mathrm{d}(y, \mathbf{K}) := \inf_{z\in\mathbf{K}} \|z - y\|_V,$$

and for two closed, non-empty, and convex sets $\mathbf{K}_1, \mathbf{K}_2$ we define the excess function e as

$$\mathrm{e}(\mathbf{K}_1, \mathbf{K}_2) := \sup_{z\in\mathbf{K}_1} \mathrm{d}(z, \mathbf{K}_2).$$

For any $\rho \ge 0$, the ρ-Hausdorff distance between $\mathbf{K}_1$ and $\mathbf{K}_2$ is given by

$$\mathrm{haus}_\rho(\mathbf{K}_1, \mathbf{K}_2) := \sup(\mathrm{e}(\mathbf{K}_1^\rho, \mathbf{K}_2), (\mathrm{e}(\mathbf{K}_2^\rho, \mathbf{K}_1)),$$

where $\mathbf{K}_i^\rho := \mathbf{K}_i \cap \rho B$, $i = 1, 2$, and B is the open unit ball centered at zero. Then, we have (see [6, Proposition 5.3]) the following.

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Theorem 3.2 (Attouch–Wets) *Let V be a Hilbert space and $\mathbf{K}_1, \mathbf{K}_2$ any two closed, convex, non-empty subsets of V. For $y_0 \in V$, we have that*

$$\|\mathrm{P}_{\mathbf{K}_1}(y_0) - \mathrm{P}_{\mathbf{K}_2}(y_0)\|_V \leq \rho^{1/2}\mathrm{haus}_\rho(\mathbf{K}_1, \mathbf{K}_2)^{1/2}$$

for $\rho := \|y_0\| + \mathrm{d}(y_0, \mathbf{K}_1) + \mathrm{d}(y_0, \mathbf{K}_2)$.

The $1/2$ exponent in the right hand side expression is optimal, and examples (even in finite dimensions) can be found where equality holds. Additionally, in Banach spaces like $L^p(\Omega)$ or $\ell^p(\mathbb{N})$, the exponent degrades even further: it is $1/p$ if $2 < p < +\infty$ and $1/p'$ if $1 < p < 2$ where p' is the Hölder conjugate of p.

In order to understand how this result fully translates into our class of maps $y \mapsto \mathbf{K}(y)$, consider the following example. Let $\Omega = (0,1)$ and $V = \{v \in H^1(\Omega) : v(0) = 0\}$ with norm $\|v\|_V^2 := \int_\Omega |v'|^2\,\mathrm{d}x$, where v' stands for the weak derivative of $v : \Omega \to \mathbb{R}$.

Suppose that $\mathbf{K}_i := \{v \in V : |\nabla v| \leq \phi_i\}$ with $\phi_2 > \phi_1 > 0$ constants. Then, if $v_i \in \mathbf{K}_i$ for $i = 1,2$, we have

$$\int_\Omega |v_2' - v_1'|^2\,\mathrm{d}x \geq \int_{\{v_2' \geq \phi_1\}} |v_2' - \phi_1|^2\,\mathrm{d}x + \int_{\{v_2' \leq -\phi_1\}} |v_2' + \phi_1|^2\,\mathrm{d}x. \tag{7}$$

Define $\tilde{v}_1(x) = \int_0^x F(\phi_1, v_2'(s))\,\mathrm{d}s$, where

$$F(\phi_1, t) := \begin{cases} \min(\phi_1, t), & t \geq 0, \\ \max(-\phi_1, t), & t < 0. \end{cases}$$

This implies that $\tilde{v}_1$ is bounded and $\tilde{v}_1' = F(\phi_1, v_2')$ in the sense of distributions, so that $\tilde{v}_1 \in H^1(\Omega)$, and in particular $\tilde{v}_1 \in V$; note that $\tilde{v}_1(0) = \lim_{x \downarrow 0} \tilde{v}_1(x) = 0$. Additionally,

$$\int_\Omega |v_2' - \tilde{v}_1'|^2\,\mathrm{d}x = \int_{\{v_2' \geq \phi_1\}} |v_2' - \phi_1|^2\,\mathrm{d}x + \int_{\{v_2' \leq -\phi_1\}} |v_2' + \phi_1|^2\,\mathrm{d}x,$$

so by (7), we have that

$$\mathrm{d}(v_2, \mathbf{K}_1)^2 = \inf_{v_1 \in \mathbf{K}_1} \int_\Omega |v_2' - v_1'|^2\,\mathrm{d}x = \int_{\{v_2' \geq \phi_1\}} |v_2' - \phi_1|^2\,\mathrm{d}x + \int_{\{v_2' \leq -\phi_1\}} |v_2' + \phi_1|^2\,\mathrm{d}x.$$

Since $-\phi_2 \leq v_2' \leq \phi_2$, for any $v_2 \in \mathbf{K}_2$ we have the bound

$$\mathrm{d}(v_2, \mathbf{K}_1)^2 \leq \int_\Omega |\phi_2 - \phi_1|^2\,\mathrm{d}x.$$

Further, if we choose $\tilde{v}_2(x) := \phi_2 x$, we have $\mathrm{d}(\tilde{v}_2, \mathbf{K}_1)^2 = \int_\Omega |\phi_2 - \phi_1|^2\,\mathrm{d}x$. Therefore

$$\mathrm{e}(\mathbf{K}_2, \mathbf{K}_1) = \sup_{v_2 \in \mathbf{K}_2} \mathrm{d}(v_2, \mathbf{K}_1) = \left(\int_\Omega |\phi_2 - \phi_1|^2\,\mathrm{d}x\right)^{1/2} = |\phi_2 - \phi_1|.$$

Also, since $\mathbf{K}_1 \subset \mathbf{K}_2$, $\mathrm{d}(v_1, \mathbf{K}_2) = 0$ for any $v_1 \in \mathbf{K}_1$ and hence $\mathrm{e}(\mathbf{K}_1, \mathbf{K}_2) = 0$. Thus, for sufficiently large $\rho > 0$, we have $\mathrm{haus}_\rho(\mathbf{K}_1, \mathbf{K}_2) = |\phi_2 - \phi_1|$. This establishes that if $\Phi\colon V \to \mathbb{R}$ is Lipschitz, then

$$\|\mathrm{P}_{\mathbf{K}(y)}(y_0) - \mathrm{P}_{\mathbf{K}(w)}(y_0)\|_V \leq \eta \|y - w\|_V^{1/2},$$

for some $\eta > 0$. Note however, that in this setting it is indeed possible to obtain a contraction for the map T; see section 3.1.2.

3.3 Order approaches: solution methods for $\mathsf{m}(f)$ and $\mathsf{M}(f)$

We consider the Gelfand triple (V,H,V') and the framework of section 2.2 including the assumptions on $A \in \mathscr{L}(V,V')$ and Φ. Then the map T is increasing and on the interval of sub- and supersolutions $[\underline{y},\overline{y}] = [A^{-1}f_{\min}, A^{-1}f_{\max}]$, there exists a minimal and a maximal solution to $(\mathrm{P_{QVI}})$, denoted $\mathsf{m}(f)$ and $\mathsf{M}(f)$, respectively. We follow a similar approach as in [15].

Consider the iterations

$$\begin{aligned} m_{n+1} &:= T(m_n), \quad m_0 := \underline{y}, \quad \text{and} \\ M_{n+1} &:= T(M_n), \quad M_0 := \overline{y}, \quad \text{for } n = 0,1,\dots \end{aligned}$$

Since $\underline{y} \le T(\underline{y})$, $T(\overline{y}) \le \overline{y}$, and $\underline{y} \le \overline{y}$, the fact that T is increasing implies that $m_n \le m_{n+1}$ and $M_{n+1} \le M_n$, and additionally $m_n, M_n \in [\underline{y},\overline{y}]$.

It can be proven than $\{m_n\}$ and $\{M_n\}$ are Cauchy sequences in H, and since they are also bounded in V, we obtain

$$m_n \uparrow m^*, \quad M_n \downarrow M^*, \qquad \text{in } H, \qquad \text{and} \qquad m_n \rightharpoonup m^*, \quad M_n \rightharpoonup M^*, \quad \text{in } V.$$

Note that $M^* \le \Phi(M_{n-1})$ for all $n \in \mathbb{N}$ so that

$$c\|M_n - M^*\|_V^2 \le \langle AM_n - AM^*, M_n - M^* \rangle \le \langle f - AM^*, M_n - M^* \rangle$$

by the fact that $M_n = S(f, \mathbf{K}(M_{n-1}))$, and hence $M_n \to M^*$ in V. Further, provided that $\Phi\colon H \to H$ is continuous, it is not hard to prove that M^* is a solution to $(\mathrm{P_{QVI}})$: from $M^* \le \Phi(M_{n-1})$, we have that $M^* \le \Phi(M^*)$, and for any $v \le \Phi(M^*)$, we have $v \le \Phi(M_{n-1})$ for any $n \in \mathbb{N}$. Hence,

$$\langle AM^* - f, v - M^* \rangle = \lim_{n\to\infty} \langle AM_n - f, v - M_n \rangle \ge 0,$$

i.e., $M^* = S(f, \mathbf{K}(M^*))$. Since $\mathsf{M}(f)$ is the maximum solution to $(\mathrm{P_{QVI}})$ on $[\underline{y},\overline{y}]$, $M^* \le \mathsf{M}(f)$. Further, since $\mathsf{M}(f) \le \overline{y}$, by repeated iteration of T on the previous inequality we have that $\mathsf{M}(f) \le M^*$, i.e., $\mathsf{M}(f) = M^*$.

In order to prove that $m^* = \mathsf{m}(f)$, additional assumptions are required. Let $\Phi\colon V \to V$ be completely continuous. Then $v_n := \min(m^*, \Phi(m_{n-1}))$ satisfies $v_n \to m^*$ in V and $v_n \le \Phi(m_{n-1})$. Hence,

$$c\|m_n - v_n\|_V^2 \le \langle Am_n - Av_n, m_n - v_n \rangle \le \langle f - Av_n, m_n - v_n \rangle,$$

where we have used that $m_n = S(f, \mathbf{K}(m_{n-1}))$. Thus, $m_n \to m^*$ in V. From $m_n \le \Phi(m_{n-1})$, and since strong convergence in H preserves order, we have $m^* \le \Phi(m^*)$. Choose $v \le \Phi(m^*)$ arbitrary and define $v_n := \min(v, \Phi(m_{n-1}))$, so that $v_n \to m^*$ in V and $v_n \le \Phi(m_{n-1})$. Then

$$\langle Am^* - f, v - m^* \rangle = \lim_{n\to\infty} \langle Am_n - f, v_n - m_n \rangle \ge 0.$$

That is, m^* is a solution to $(\mathrm{P_{QVI}})$ within $[\underline{y},\overline{y}]$. Hence, by definition of $\mathsf{m}(f)$, we have $\mathsf{m}(f) \le m^*$, and from $\underline{y} \le \mathsf{m}(f)$ and the consecutive iteration of T on the previous inequality, we have $m^* \le \mathsf{m}(f)$, i.e., $m^* = \mathsf{m}(f)$. Overall, we have the following result.

Proposition 3.1 *In addition to the assumptions for Φ in section 2.2, suppose that $\Phi\colon V \to V$ is completely continuous. Then $m_n \uparrow \mathsf{m}(f)$ and $M_n \downarrow \mathsf{M}(f)$ in H and $m_n \to \mathsf{m}(f)$ and $M_n \to \mathsf{M}(f)$ in V.*

Open problems. *The speed of convergence of $\{m_n\}$ and $\{M_n\}$ is, in general, slower than linear. This hinders their applicability when addressing large scale problems, or when considering optimization problems involving $\mathsf{m}(f)$ and $\mathsf{M}(f)$, as in section 4. It is an open question whether it is possible to accelerate such iterations by combining them with intermediate steps. Additionally, it is open wether linearly convergent methods can be designed in general when the solution is non-unique.*

3.4 Regularization methods

3.4.1 Extended Moreau–Yosida and Semismooth Newton

It is convenient to consider regularizations of QVIs by smoothing. The type of regularization or smoothing that we consider in this section consists of approximating the QVI in question by a sequence of parameter-dependent PDEs. Regularization methods are useful for numerical purposes as well as for theoretical efforts. For example, they can be used to prove fundamental results such as existence of solutions as well as to derive stationarity conditions for optimal control problems with QVI constraints[1], which is a subject of work under preparation by the authors. Moreover, even for VIs, obtaining mesh independence requires regularization.

Obstacle case

For simplicity, we consider $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. In this section we present some results on the Moreau–Yosida regularization of the obstacle type ($\mathrm{P_{QVI}}$) given by the nonlinear PDE

$$F(y) := Ay - f + \frac{1}{\beta}(y - \Phi(y))^+ = 0 \tag{8}$$

for $\beta > 0$. Under suitable assumptions it is expected that as $\beta \downarrow 0$, the sequence of solutions y^*_β converges to the solution of ($\mathrm{P_{QVI}}$). In fact, if $\Phi\colon V \to V$ is increasing and completely continuous with $\Phi(0) \geq 0$ and $f \in V'_+$, then $\{y^*_{\beta_n}\}$ has a subsequence that converges in V to a solution of ($\mathrm{P_{QVI}}$), for any $\beta_n \downarrow 0$.

Focusing on (8), we consider $y_0 \in \tilde{V} \subset V$, and the Newton iteration

$$y_{k+1} = y_k - G_F(y_k)^{-1} F(y_k), \qquad k = 0, 1, 2, \ldots \tag{9}$$

where $G_F(y) \in \mathscr{L}(V, V')$ is a (presumably invertible) Newton derivative of F [36], which is defined to satisfy

$$\lim_{h \to 0} \frac{\|F(y+h) - F(y) - G_F(y+h)h\|_{V'}}{\|h\|_V} = 0.$$

It is know that $(\cdot)^+\colon L^p(\Omega) \to L^2(\Omega)$ is Newton differentiable for any $p > 2$ with Newton derivative $G_{\max}(y) = \mathrm{Heaviside}(y)$. Suppose that $\Phi\colon V \to L^q(\Omega)$ is Fréchet differentiable for some $q \geq p$, then we have (see [45, Lemma 8.15]) that $G_F(y) \in \mathscr{L}(V, V')$ is given by

$$G_F(y)h = Ah + \frac{1}{\beta} G_{\max}(y - \Phi(y))(I - \Phi'(y))h.$$

Suppose that $(\chi_{\Omega_0}\Phi'(y)h, h) \leq (\chi_{\Omega_0}h, h)$ for any $\Omega_0 \subset \Omega$ and for all $y, h \in V$. Then $\langle G_F(y)h, h\rangle \geq \tilde{c}\|h\|_V^2$ for some $\tilde{c} > 0$ so that $\|G_F(y)^{-1}\|_{\mathscr{L}(V',V)} \leq 1/\tilde{c}$ and hence (9) converges superlinearly to the solution y^*_β of (8), provided that $\|y_0 - y^*_\beta\|_V$ is sufficiently small; see [36–38, 45].

Example on thermoforming. The production of plastic parts is in general done by thermoforming. In this procedure, a plastic sheet is heated to its pliable temperature and then forced via air pressure (positive or negative) towards a mold, commonly made of metal, and involving some cooling mechanism. Such a manufacturing process involves several scales: it is used for microfluidic structures, plastic cups, and large parts in the automotive industry.

[1] Naturally optimality conditions obtained through regularization will not be as strong as those potentially obtained through using the directional differentiability of the QVI solution mapping, see section 5.

We consider the following time-asymptotic behaviour of the thermoforming process leading to an elliptic problem. We let a plastic membrane y lie over the domain Ω, and let the temperature of the membrane be constant (this simplification frees us from considering changing rheological properties of the heated membrane).

The mathematical problem is then given by: Find $(y,\Phi,T) \in V \times V \times W$ such that

$$
\begin{aligned}
y \leq \Phi, \quad \langle Ay - f, y - v\rangle &\leq 0, && \forall v \in V : v \leq \Phi, && (10)\\
\langle kT - \Delta T, w\rangle &= (g(\Phi - u), w) && \forall w \in W, && (11)\\
\Phi &= \Phi_0 + LT && \text{in } V, && (12)
\end{aligned}
$$

where $f \in H_+$, $k > 0$ is a constant, $\Phi_0 \in V$ is the desired mold, and $L\colon W \to V$ is a bounded linear operator such that

$$\text{for every } \Omega_0 \subset \Omega, \text{ if } u \leq v \text{ a.e. on } \Omega_0 \text{ then } Lu \leq Lv \text{ a.e. on } \Omega_0,$$

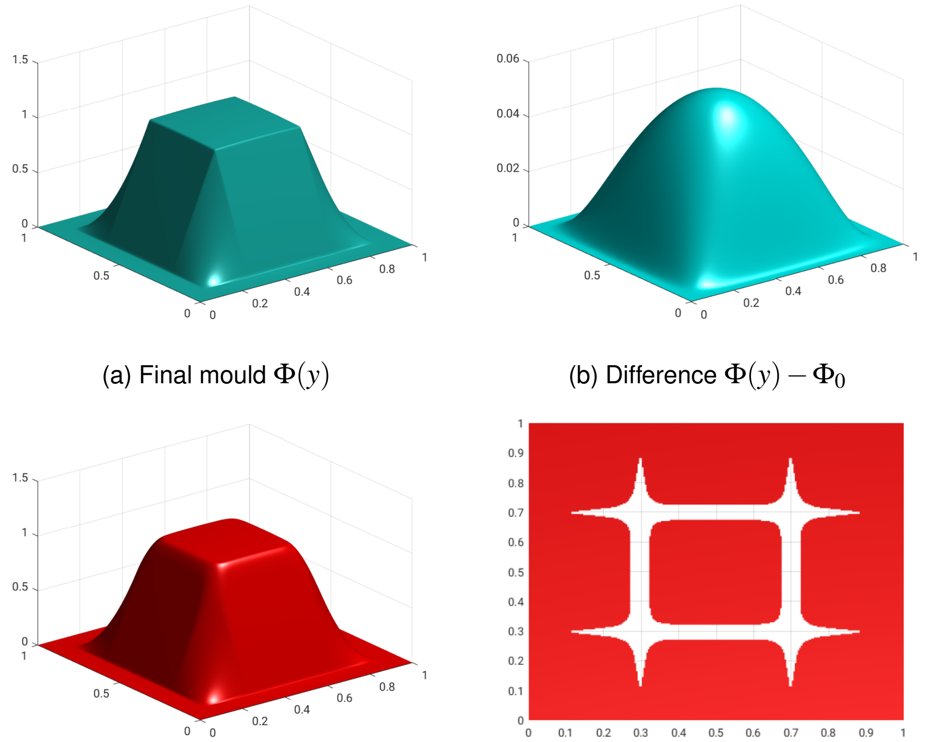
and $g\colon \mathbb{R} \to \mathbb{R}$ is decreasing with $g(0) = M > 0$ a constant, $0 \leq g \leq M$ and g' bounded.

The above problem can be equivalently formulated as problem ($\mathrm{P}_{\mathrm{QVI}}$) where $\Phi\colon W \to V$ is defined as follows. Let $v \in W$ and consider the problem: Find $\phi \in V$ such that

$$
\begin{aligned}
\langle kT - \Delta T, w\rangle &= (g(\phi - v), w) && \forall w \in W, && (13)\\
\phi &= \Phi_0 + LT && \text{in } V. && (14)
\end{aligned}
$$

We define $\Phi(v) = \phi$.

In Figure 1, we see the membrane y, the obstacle $\Phi(y)$, the coincidence set, and the difference $\Phi(y) - \Phi_0$, all computed with the semismooth Newton method described above for β sufficiently large (full details of the analysis and numerical implementation of the models presented here can be found in section 6 of [3]).



(a) Final mould $\Phi(y)$ (b) Difference $\Phi(y)-\Phi_0$

(c) Membrane y (d) Coincidence set $\{y=\Phi(y)\}$ (in red)

Figure 1: Results for the thermoforming example

Gradient case

We consider here $V=W_0^{1,p}(\Omega)$ and $H=L^2(\Omega)$. The type of regularization used in (8) is not amenable for direct application in the gradient case. In fact, provided A is symmetric, one can consider the minimization problem

$$\min_{y\in V}\frac{1}{2}\langle Ay,y\rangle-\langle f,y\rangle+\frac{1}{\beta}\left\|(|\nabla y|-\Phi(y))^+\right\|_H^2 \tag{15}$$

associated to the QVI with the gradient constraint. In connection with (15), it was proven in [40, Theorem 3.2] that there is a sequence of β such that the associated solutions to the penalized minimization problems converge to the solution of the minimization problem $\min_{y\in V}\frac{1}{2}\langle Ay,y\rangle-\langle f,y\rangle$ subject to $|\nabla y|\le\Phi(y)$ a.e in Ω, which is not in general a solution of ($\mathrm{P}_{\mathrm{QVI}}$). This fact is in sharp contrast to the VI setting: in fact, if $\Phi(y)$ is replaced by $\Phi(w)$ in (15) for some $w\in V$, then the problem is suitable for a semismooth Newton approach and the sequence of solutions $\{y_\beta(w)\}_\beta$ converges, as $\beta\downarrow 0$, in V to $y^*=S(f,\mathbf{K}(w))$. In this case, we have that $y_\beta(w)\in V$ satisfies

$$\left.\begin{array}{l}
F(y):=\langle A(y),v\rangle-\langle f,v\rangle+\frac{1}{\beta}((|\nabla y|-\Phi(w))^+,q(\cdot)\nabla v)=0\\
q(x)\in\begin{cases}\frac{\nabla y}{|\nabla y|}(x), & \text{if } |\nabla y(x)|>0\\ \bar{B}_1(0)^N, & \text{otherwise,}\end{cases}\\
\text{for all } v\in V,
\end{array}\right\} \tag{16}$$

where $\bar{B}_1(0)^N$ denotes the closed unit ball in $\mathbb{R}^N$.

The application of the semismooth Newton method for the resolution of $F(y) = 0$ in this case has several subtleties. Specifically, the existence of a Newton derivative of the map $y \mapsto \mathscr{P}(y) := -\mathrm{div} q(\cdot)^{\mathrm{T}}((|\nabla y| - \Phi(w))^{+}$ requires a delicate interplay of the domain and image spaces. In contrast to $(\cdot)^{+} \colon L^p(\Omega) \to L^2(\Omega)$, which is Newton differentiable for any $p > 2$, the aforementioned map is Newton differentiable when considered as $\mathscr{P} : W_0^{1,p}(\Omega) \to W^{-1,s}(\Omega)$, with $3 \leq 3s \leq p < \infty$; see [40].

3.5 Gerhardt-type regularization for the gradient case

For simplicity we consider the gradient case where $A = -\Delta$ is simply the Laplacian. We briefly discuss here an extension of a technique introduced by Gerhardt [30] which was developed by Rodrigues, Santos and collaborators in a series of papers; see [10, 11, 61, 68].

One way to regularize problem ($\mathrm{P}_{\mathrm{QVI}}$) in the case described above is through the PDE

$$-\nabla \cdot (g_\varepsilon(|\nabla y|^2 - \Phi^2(y))\nabla y) - f = 0 \tag{17}$$

where $g_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ is a bounded non-decreasing function which is twice continuously differentiable with

$$g_\varepsilon(t) = \begin{cases} 1 & : t \leq 0, \\ e^{t/\varepsilon} & : \varepsilon \leq t \leq \frac{1}{\varepsilon} - \varepsilon, \\ e^{1/\varepsilon^2} & : t \geq \frac{1}{\varepsilon}, \end{cases} \tag{18}$$

for $\varepsilon > 0$. Formally, it can be be thought of as an approximation to

$$g_0(t) = \begin{cases} 1 & : t \leq 0, \\ \infty & : t > 0. \end{cases}$$

This suggests that in the limiting process (as $\varepsilon \to 0$) for the nonlinear term not to blow up, the argument inside the regularization function needs to be non-positive, which of course then retrieves the gradient constraint. This type of regularization was first introduced by Gerhardt [30] with the aim of approximating the solution to an elliptic minimization problem, and the specific form (18) was used in [61,67] to tackle parabolic variational inequality problems. See also [13,69]. The function g_ε satisfies the useful monotonicity property [13]

$$(g_\varepsilon(|x|^2 - a)x - g_\varepsilon(|y|^2 - a)y)(x - y) \geq 0$$

which allows one to pass to the limit in the weak formulation of (17) after having obtained uniform estimates. Rigorous details of this can be found in the cited works.

This type of regularization, though powerful in the theoretical setting, has not been proven useful yet in the development of solution algorithms. In fact, if we formulate (17) as $F(y) = 0$ and try to identify a Newton derivative (as done in the previous section), we face differentiating the highest order terms of the associated nonlinear differential operator, a complex task in its own right. Furthermore, the Newton-type iterations would require, in the case of discretization by finite elements, a time consuming reassembling of the stiffness matrix in each iteration.

3.6 Drawbacks of the iteration $y_{n+1} = T(y_n)$

Since problem (2) is suitable for numerical resolution via diverse methods, a first approach for computing fixed points of T is to consider the iteration

$$y_{n+1} = T(y_n), \qquad n = 0, 1, \dots,$$

with $y_0 \in V$ given.

The properties of A determine that the sequence $\{u_n\}$ is bounded in V and hence it contains weakly convergent subsequences. Additionally, suppose that sufficient properties of Φ are available so that $\mathbf{K}(v_n) \to \mathbf{K}(v)$ in the sense of Mosco if $v_n \rightharpoonup v$ in V. Then it follows that $T\colon V \to V$ is completely continuous: if $v_n \rightharpoonup v$, then $T(v_n) \to T(v)$ in V.

This seemingly amenable circumstance described above leads to the following erroneous argument that is common in the literature: "*Denote also by $\{y_n\}$ a weakly convergent subsequence of $\{y_n\}$ with limit y^*. Then taking the limit on both sides of $y_{n+1} = T(y_n)$, we observe that y^* is a fixed point of* T". The mistake clearly lies in assuming that if $y_{n_k} \rightharpoonup y^*$, then $\{y_{n_k+1}\}$ has the same weak limit. In particular what this attemps to show is that the compactness properties of T determine that the sets of weak and strong accumulation points of $\{y_n\}$ (denoted as $\mathscr{A}$) are identical, and if $y \in \mathscr{A}$, then $T(y) \in \mathscr{A}$.

Since $T(\mathscr{A}) \subset \mathscr{A}$, we can try to extend the digression further and study the possibility of finding a fixed point since we have now a T-invariant set. If $\mathscr{A}$ can be proven to be convex (it is usually not), then T has a fixed point in $\mathscr{A}$ via Schauder's fixed point. The alternative is to consider the search of a fixed point in the closed convex hull of $\mathscr{A}$ denoted by $\overline{\mathrm{co}}\,\mathscr{A}$. If $T(\overline{\mathrm{co}}\,\mathscr{A}) \subset \overline{\mathrm{co}}\,T(\mathscr{A})$ holds true, then $T(\overline{\mathrm{co}}\,\mathscr{A}) \subset \overline{\mathrm{co}}\,\mathscr{A}$ and Schauder's fixed point theorem can be used to deduce that T has a fixed point in $\overline{\mathrm{co}}\,\mathscr{A}$. However, for obstacle type problems, if Φ is concave, the map T is too (see [15]), so that $T(\mathrm{co}\,\mathscr{A}) \geq \mathrm{co}\,T(\mathscr{A})$.

4 Optimal control problems

There exist several applications for optimization problems where the QVI is a constraint. Then, often solutions of the QVI are controlled in such a way that they are close to some desired state. These types of problems have been almost completely neglected in the literature. An instance of such a problem is

Problem $(\mathbb{P})$:

$$\begin{aligned} &\text{minimize} \quad J(y,f) \text{ over } (y,f) \in V \times U, \\ &\text{subject to} \quad f \in U_{\mathrm{ad}} \subset U \subset V', \text{ and } y \text{ solves } (\mathrm{P}_{\mathrm{QVI}}), \end{aligned} \tag{$\mathbb{P}$}$$

where $J\colon V \times U \to \mathbb{R}$ is weakly lower semicontinuous and U_{ad} is compact in V'. Note that if $\mathbf{K}(v_n) \xrightarrow{\mathrm{M}} \mathbf{K}(v)$ whenever $v_n \rightharpoonup v$ in V, then problem $(\mathbb{P})$ has a solution: Indeed, let $\{(y_n, f_n)\}$ be an infimizing sequence. Then, there exists a subsequence, denoted also as $\{(y_n, f_n)\}$, such that $y_n \rightharpoonup y^*$ in V, $f_n \rightharpoonup f^*$ in U and $f_n \to f^*$ in V'. We have that $y_n = S(f_n, \mathbf{K}(y_n))$ and $y^* = S(f^*, \mathbf{K}(y^*))$ by taking limits on both sides, and hence $\lim J(y_n, f_n) = J(y^*, f^*)$ so that (y^*, f^*) is a minimizer of the problem.

The literature on such problems is scarce; see [1, 20] for exceptions. Further, it falls short in tackling the real problems in the QVI setting. The solution set of the QVI is in general not a singleton, and in case of industrial applications it is of interest to control the entire solution set. In view of this, we have the following open questions.

Open problems. *In the QVI context, it is sometimes important to control the full solution set* $\mathbf{Q}(f)$ *on a certain interval of interest* $[\underline{y},\overline{y}]$. *We consider the Gelfand triple setting of section 2.2. A possible formulation for such control problems is as follows:*

Problem $(\tilde{\mathbb{P}})$*:*

$$\begin{aligned}&\text{minimize } J(\mathbf{O},f) := J_1(T_{\sup}(\mathbf{O}),T_{\inf}(\mathbf{O}),f)\\&\text{over} \quad (\mathbf{O},f)\in 2^H\times U,\\&\text{subject to } f\in U_{ad},\\&\qquad\qquad y\in\mathbf{O},\quad \mathbf{O}=\{z\in V : z \textit{ solves } (\mathrm{P}_{\mathrm{QVI}})\cap[\underline{y},\overline{y}]\}.\end{aligned} \tag{$\tilde{\mathbb{P}}$}$$

In the above problem we consider $J_1\colon H\times H\times U\to\mathbb{R}$ *and for* $\underline{y},\overline{y}\in H$ *we define the set map* $T_{\sup}$

$$T_{\sup}(\mathbf{O}) := \begin{cases}\sup_{z\in\mathbf{O}\cap[\underline{y},\overline{y}]} z, & \mathbf{O}\cap[\underline{y},\overline{y}]\neq\emptyset\ ;\\ \underline{y}, & \textit{otherwise.}\end{cases}$$

The map $T_{\inf}$ *defined analogously as*

$$T_{\inf}(\mathbf{O}) := \begin{cases}\inf_{z\in\mathbf{O}\cap[\underline{y},\overline{y}]} z, & \mathbf{O}\cap[\underline{y},\overline{y}]\neq\emptyset\ ;\\ \overline{y}, & \textit{otherwise.}\end{cases}$$

As explained in section 2.2, the supremum of an arbitrary subset of H *that is bounded above (in the order) is also correctly defined since* H *is Dedekind complete, which shows that* $T_{\inf}$ *and* $T_{\sup}$ *are well defined in our setting.*

Recall the framework of section 2.2 where $\underline{y}$ *and* $\overline{y}$ *are respectively sub- and supersolutions of the map* $T(\cdot)=S(f,\mathbf{K}(\cdot))$. *Then the reduced version of problem* $(\tilde{\mathbb{P}})$ *is formulated in terms of the operators* m *and* M *as*

$$\begin{aligned}&\text{minimize } J_1(\mathsf{M}(f),\mathsf{m}(f),f)\\&\text{subject to } f\in U_{ad}.\end{aligned} \tag{$\tilde{\mathbb{P}}_{\mathrm{red}}$}$$

An important example is when it is required to force the solution set to be a singleton and the element in question to be close to some desired state y_d. *Here, a possible choice for* J_1 *is given by*

$$J_1(\mathsf{M}(f),\mathsf{m}(f),f) = \frac{1}{2}\int_\Omega|\mathsf{M}(f)-\mathsf{m}(f)|^2\,\mathrm{d}x + \frac{\sigma}{2}\int_\Omega|y_d-\mathsf{m}(f)|^2\,\mathrm{d}x.$$

To the best of our knowledge, problem $(\tilde{\mathbb{P}})$ *(and its reduced version) has not been considered in the literature, and it is a topic of active research by the present authors. Important (and currently still open) subtasks for analyzing the above control problem are (i) the study of stability properties of the maps* $f\mapsto\mathsf{M}(f),\mathsf{m}(f)$ *and (ii) their (generalized) differentiability properties. While (i) typically helps to establish existence of a solution to the optimization problem, (ii) allows for suitable stationarity conditions characterizing solutions.*

5 Differentiability

We consider in this section the differential stability of the solution map associated to $(\mathrm{P}_{\mathrm{QVI}})$, in particular, the mapping taking the source term into the set of the solutions. Showing that this map is differentiable (in some sense) is not only an interesting analytical task in its own right but is also of use for optimal control, numerics and applications.

The corresponding differentiability study for variational inequalities has been thoroughly investigated [33, 59, 74]. Let us set the scene and outline this theory first before moving on to QVIs.

Let X be a locally compact topological space which is countable at infinity with ξ a Radon measure on X. Suppose $V \subset L^2(X;\xi) =: H$ is a Hilbert space with the embedding continuous and dense and such that $|u| \in V$ whenever $u \in V$, and let $A\colon V \to V'$ be now a linear operator satisfying the boundedness, coercivity and T-monotonicity properties from before, i.e., (A1), (A2), and (A3). The pair (V,A) falls into the class of *positivity preserving coercive forms* with respect to $L^2(X;\xi)$ [16, 58]. We further assume that

$$V \cap C_c(X) \subset C_c(X) \quad \text{and} \quad V \cap C_c(X) \subset V \quad \text{are dense embeddings,} \tag{19}$$

thus (V,a) is a *regular* form [27, §1.1] [16, §2][2]. This framework allows us to define the notions of capacity, quasi-continuity and related objects, see [59, §3] and [33, §3]. Several concrete examples of V and A are given in [59, §3] and [3, §1.2].

Given an obstacle $\phi \in V_+$, we define the set

$$\mathbf{K} := \{w \in V : w \leq \phi\},$$

and given a source term $f \in V'$, we make an abuse of notation here and define by $S\colon V' \to V$ the mapping $S(f) := S(f,\mathbf{K})$ with the latter defined in (2). It is useful to introduce the well known notions of the *tangent cone* and the *critical cone* associated to $\mathbf{K}$, given respectively by

$$T_{\mathbf{K}}(y) := \{\varphi \in V : \varphi \leq 0 \text{ q.e. on } \{y = \phi\}\} \text{ and } \mathscr{K}_{\mathbf{K}}(y) := T_{\mathbf{K}}(y) \cap [f - Ay]^{\perp}. \tag{20}$$

The coincidence set appearing in the tangent cone is of course calculated over X. This is worth emphasis since for example if V is chosen to be the Sobolev space $H^1(\Omega)$ on a bounded Lipschitz domain Ω, then X should be $\bar{\Omega}$, the closure of the domain, and not Ω itself; see [3, §1.2].

The following result of Mignot tells us that the mapping S is directionally differentiable.

Theorem 5.1 (Theorem 3.3 of [59]) *Given $f \in V'$ and $d \in V'$, there exists a function $S'(f)(d) \in V$ such that*

$$S(f + td) = S(f) + tS'(f)(d) + o(t) \quad \forall t > 0$$

holds where $t^{-1}o(t) \to 0$ as $t \to 0^+$ in V and $\delta := S'(f)(d)$ satisfies the VI

$$\delta \in \mathscr{K}_{\mathbf{K}}(y) : \langle A\delta - d, v - \delta \rangle \geq 0 \quad \forall v \in \mathscr{K}_{\mathbf{K}}(y), \text{ where } y = S(f).$$

The directional derivative $\delta = \delta(d)$ is positively homogeneous in d.

[2] A space V under all of the previous assumptions except the second density assumption in (19) is referred to by Mignot in [59] as a ‘Dirichlet space’ — this is rather inconsistent with the modern literature [27] where Dirichlet spaces and Dirichlet forms are defined differently (see [27, §1.1]).

In [44], the authors essentially extended the results of Mignot to a more general setting and turned the question of directional differentiability for VIs with more general constraint sets (than those of obstacle type) into a geometric question of the *polyhedricity* of the underlying constraint set, and more details and background can be found in the cited text.

One says that *strict complementarity* holds if the critical cone simplifies to the linear subspace

$$\mathscr{K}_{\mathbf{K}}(y) = \mathscr{S}_{\mathbf{K}}(y) := \{\varphi \in V : \varphi = 0 \text{ q.e. on } \{y = \phi\}\}. \tag{21}$$

In this case, the VI satisfied by δ simplifies to a variational equality due to the relaxation of constraints on the test functions for the inequality. It is not hard to see that, at least formally, strict complementarity arises when the biactive set $\{Ay - f = 0\} \cap \{y - \phi = 0\}$ is empty; see [28, 29] for some technical details regarding biactivity that include its proper definition under low regularity of y and f. Under strict complementarity, the derivative in Theorem 5.1 is in fact a Gâteaux derivative as the next result shows.

Theorem 5.2 (Theorem 3.4 of [59]) *In the context of Theorem 5.1, if strict complementarity holds, then the derivative* α *satisfies*

$$\alpha \in \mathscr{S}_{\mathbf{K}}(y) : \langle A\delta - d, v - \delta \rangle = 0 \quad \forall v \in \mathscr{S}_{\mathbf{K}}(y).$$

In this case, $\delta = \delta(d)$ *is linear in* d.

5.1 Directional differentiability for QVIs

To formulate the QVI case, let $\Phi \colon V \to V$ be increasing with $\Phi(0) \geq 0$. Given $f \in V'$, consider ($\mathrm{P}_{\mathrm{QVI}}$) in the obstacle case (i.e., $\psi \circ G \equiv \mathrm{id}$):

$$y \in \mathbf{K}(y) : \quad \langle Ay - f, v - y \rangle \geq 0 \quad \forall v \in \mathbf{K}(y). \tag{22}$$

We consider $\mathbf{Q} \colon V'_+ \rightrightarrows V$, the multi-valued solution mapping taking $f \mapsto y$. To show that this map is directionally differentiable (in some sense), the obvious idea that springs to mind is to rewrite (22) by transforming the obstacle onto the source term and then to apply Mignot's theory. Indeed, the inequality implies that the quantity $\hat{y} := (\mathrm{id} - \Phi)y$ solves

$$\hat{y} \in \mathbf{K}_0 : \langle A(\mathrm{id} - \Phi)^{-1}\hat{y} - f, \phi - \hat{y} \rangle \geq 0 \quad \forall \phi \in \mathbf{K}_0,$$

with $\mathbf{K}_0 := \{w \in V : w \leq 0\}$; however, in general, the elliptic operator $A(\mathrm{id} - \Phi)^{-1}$ is not linear, coercive nor T-monotone, so the VI theory is not applicable and a different approach is needed.

The idea in [3] is the following: approximate the QVI solution $q(t) \in \mathbf{Q}(f + td)$ by a sequence $q_n(t)$ of solutions of VIs (each of which by definition has a explicit obstacle), obtain suitable differential formulae for those VIs and then pass to the limit to (hopefully) obtain an expansion formula relating elements of $\mathbf{Q}(f + td)$ to $\mathbf{Q}(f)$. There are some delicacies in this procedure:

1 **derivation of the expansion formulae** for the above-mentioned VI iterates $q_n(t)$; they must relate $q(t)$ to a solution $y \in \mathbf{Q}(f)$, and recursion plays a highly nonlinear role in the relationship between one iterate and the preceding iterates;

2 **obtaining uniform bounds on the directional derivatives**; even though the derivatives satisfy a VI, it requires the handling of a recurrence inequality unless some regularity is available (see [3, §4.3]);

3 **identifying the limit of the higher-order terms as a higher-order term**; this procedure involves two limits: one as $t \to 0^+$ and one as $n \to \infty$, and commutation of limits in general requires an additional uniform convergence.

The main difficulty is indeed the final point above. Although the directional derivatives and higher-order terms of the VI iterates do possess some monotonicity properties, this information unfortunately does not help as much as one may hope.

The iteration scheme alluded to above requires some further restrictions on the data f and the direction d that the derivative is taken in, and we shall outline these in the following. We assume that $f \in V'_+$ and define $\bar{y} \in V$ as the (non-negative) weak solution of the unconstrained problem $A\bar{y} = f$. In a similar fashion to $\bar{u}$, define $\bar{q}(t) \in V$ as the solution of the unconstrained problem with right hand side $f + td$: $A\bar{q}(t) = f + td$.

Since we are considering the issue of sensitivity of QVIs with (by definition) implicit obstacles defined through the mapping Φ, it is clear that further regularity is required of Φ. We introduce these further assumptions below where we state the main theorem of [3], but first let us define

$$\mathscr{K}_{\mathbf{K}(y)}(y,\alpha) := \Phi'(y)(\alpha) + \mathscr{K}_{\mathbf{K}(y)}(y)$$

which can be thought as a *translated* critical cone.[3]

Theorem 5.3 (Theorem 1.6 of [3]) *Let $f, d \in V'_+$. Given $y \in \mathbf{Q}(f) \cap [0, \bar{y}]$, assume the following:*

$(\mathrm{H1})$ *the map $\Phi\colon V \to V$ is Hadamard directionally differentiable*[4]

$(\mathrm{H2})$ *either*

$(\mathrm{H2})$*.1* $\Phi\colon V \to V$ *is completely continuous, or*

$(\mathrm{H2})$*.2* $V = H^1(\Omega), X = \overline{\Omega}$ *where Ω is a bounded Lipschitz domain, $\Phi\colon L^\infty_+(\Omega) \to L^\infty_+(\Omega)$ and is concave with $\Phi(0) \geq c > 0$, and $f, d \in L^\infty_+(\Omega)$*[5]

$(\mathrm{H3})$ *the map $\Phi'(v)\colon V \to V$ is completely continuous (for fixed $v \in V$)*

$(\mathrm{H4})$ *for any $b \in V$, $h\colon (0,T) \to V$ and $\lambda \in [0,1]$,*

$$\frac{\|\Phi'(y + tb + \lambda h(t))h(t)\|_V}{t} \to 0 \text{ as } t \to 0^+ \text{ if } \frac{h(t)}{t} \to 0 \text{ as } t \to 0^+$$

$(\mathrm{H5})$ *given $T_0 \in (0,T)$ small, if $z\colon (0,T_0) \to V$ satisfies $z(t) \to y$ as $t \to 0^+$, then*

$$\left\|\Phi'(z(t))b\right\|_V \leq C_\Phi \|b\|_V \quad \text{where } C_\Phi < \frac{1}{1 + c^{-1}C}$$

for all $t \in (0,T_0)$, where C and c are from $(\mathrm{A1})$ *and* $(\mathrm{A2})$*.*

[3] Explicitly this set is $\{\varphi \in V : \varphi \leq \Phi'(y)(w)$ q.e. on $\{y = \Phi(y)\}$ and $\langle Ay - f, \varphi - \Phi'(y)(w)\rangle = 0\}$.

[4] In fact, $(\mathrm{H1})$ can be weakened significantly by requiring Hadamard differentiability of Φ only at the point y, i.e., locally, as in assumptions $(\mathrm{H4})$ and $(\mathrm{H5})$.

[5] In this case, solutions of the QVI (22) are unique [55].

Then there exists $q(t) \in \mathbf{Q}(f+td) \cap [y, \bar{q}(t)]$ *and* $\alpha = \alpha(d) \in V_+$ *such that*

$$q(t) = y + t\alpha + o(t) \quad \forall t > 0$$

holds where $t^{-1}o(t) \to 0$ *as* $t \to 0^+$ *in* V *and* α *satisfies the QVI*

$$\alpha \in \mathscr{K}_{\mathbf{K}(y)}(y, \alpha) : \langle A\alpha - d, v - \alpha \rangle \geq 0 \quad \forall v \in \mathscr{K}_{\mathbf{K}(y)}(y, \alpha)$$

The directional derivative $\alpha = \alpha(d)$ *is positively homogeneous in* d*.*

It should be emphasized that the assumptions $(\mathrm{H4})$ and $(\mathrm{H5})$ depend on the specific function y, i.e., these are *local* conditions. The assumption $(\mathrm{H5})$ implies certain restrictions: in the case that Φ is linear, it imposes a smallness condition on the operator norm of Φ which enforces uniqueness of solutions of the QVI. However, it does not necessarily rule out the multivalued setting in the case of nonlinear Φ.

Open problems. *The result in the general multi-valued setting given in Theorem 5.3 is a differentiability result for a specific selection mechanism that associates to a function* $y \in \mathbf{Q}(f)$ *a function* $q(t) \in \mathbf{Q}(f+td)$ *(the precise mechanism is expounded in [3, §3.2.1]). A useful variant of the theorem would be to obtain the result for the mapping that selects the minimal or maximal solution to the QVI, i.e., if* $\mathsf{M}(f) \in \mathbf{Q}(f)$ *is the maximal solution of the QVI with source term* f*, is* M *directionally differentiable? A difficulty lies in the approximation scheme we use; in the proof of Theorem 5.3 we chose* $q_0 = y$*; instead we could choose* $q_0 = y_0$ *where* $0 \leq y_0 \leq \bar{y}$ *which leads to the equality*

$$q_n(t) = y_n(t) + t\hat{\alpha}_n + \hat{o}_n(t)$$

where $y_n = S(f, \mathbf{K}(y_{n-1}))$*. The main problem is in dealing with the limiting behaviour of the higher-order terms* $\hat{o}_n(t)$*, which now depends on the base point* y_n *which depends on* n*. This fact constrains us in this direction. For more details see [3, Remark 3.9].*

It is worth restating Theorem 5.3 in the case when $\mathbf{Q}\colon V_+' \rightrightarrows V$ is single-valued (i.e., the QVI problem has a unique solution).

Theorem 5.4 *Suppose* $\mathbf{Q}$ *is single-valued and let the hypotheses of Theorem 5.3 hold given* $f, d \in V_+'$*. There exists a function* $\mathbf{Q}'(f)(d) \in V_+$ *such that*

$$\mathbf{Q}(f+td) = \mathbf{Q}(f) + t\mathbf{Q}'(f)(d) + o(t) \quad \forall t > 0$$

holds where $t^{-1}o(t) \to 0$ *as* $t \to 0^+$ *in* V *and* $\mathbf{Q}'(f)(d)$ *satisfies the QVI given in Theorem 5.3.*

Similarly to Theorem 5.2, under a modification of the notion of strict complementarity, we obtain a regularity result on the directional derivative. In this setting, *strict complementarity* holds if the set $\mathscr{K}_{\mathbf{K}(y)}(y, w)$ simplifies to

$$\begin{aligned} \mathscr{K}_{\mathbf{K}(y)}(y, w) &= \mathscr{S}_{\mathbf{K}(y)}(y, w) \\ &:= \{\varphi \in V : \varphi = \Phi'(y)(w) \text{ q.e. on } \{y = \Phi(y)\}\}. \end{aligned}$$

Theorem 5.5 (Theorem 1.7 of [3]) *In the context of Theorem 5.3, if strict complementarity holds, then the derivative α satisfies*

$$\alpha \in \mathscr{S}_{\mathbf{K}(y)}(y,\alpha) : \langle A\alpha - d, \alpha - v\rangle = 0 \quad \forall v \in \mathscr{S}_{\mathbf{K}(y)}(y,\alpha).$$

In this case, if $h \mapsto \Phi'(v)(h)$ is linear, $\alpha = \alpha(d)$ satisfies $\alpha(c_1d_1 + c_2d_2) = c_1\alpha(d_1) + c_2\alpha(d_2)$ for constants $c_1, c_2 > 0$ and directions $d_1, d_2 \in V'_+$.

Naturally, we recover the results of [59] in the case where Φ is a constant mapping.

Open problems. *A focus of ongoing work by the authors is the study of optimal control problems with QVI constraints of the following type:*

Problem $(\mathbb{P}')$ *:*

$$\begin{aligned} &\textit{minimize} \quad \frac{1}{2}\|y - y_d\|_H^2 + \frac{\lambda}{2}\|f\|_U^2 \ \textit{ over } (y,f) \in V \times U, \\ &\textit{subject to} \quad f \in U_{ad} \subset U \subset V', \ \textit{ and } y \textit{ solves } (22), \end{aligned} \tag{23}$$

Here, the data y_d is a desired state and $\lambda > 0$ is a constant. Under certain assumptions on the mapping Φ and the spaces featured above, existence of an optimal control and state can be shown using relatively standard methods. Obtaining stationarity conditions that explicitly characterize the optimal control and optimal state (which would, in particular, allow for a feasible numerical resolution of the problem) is of prime importance in optimization. Typically, strong *stationarity conditions are sought and such conditions in the VI case have been obtained [60] by making use of the differentiability of the VI solution mapping, and we would like to extend this result also to the QVI case. A challenge lies in the fact that, in Theorem 5.3, differentiability (in the QVI setting) is only obtained for non-negative directions. Hence, problem* (23) *would contain pointwise a.e. bounds on the control. From [73] it is however known that obtaining strong stationarity is impossible in the VI case with such pointwise a.e. control bounds (without further restrictions on the bounds themselves). This represents a major issue. However, there are other notions of stationarity (see [39]) that could be obtained.*

6 Conclusion

We have considered a variety of key topics and we have highlighted limitations and open questions associated to QVIs of elliptic type. For the existence results we focused on compactness approaches and the lack of necessity and sufficiency results for Mosco convergence in cases other than constraint sets of obstacle type, and we also tackled some order approaches. For the simple fixed point arguments, we provided some positive results, and showed that the popular extension approaches to Lions–Stampacchia are in the best case scenario unnecessary. Additionally, we have provided some second-order solution algorithms of the semismooth Newton type. Finally, we have established some novel optimization problems that take into account the multivalued nature of the solution set of the QVI and gave an account of the newly established directional differentiability for the QVI solution map.

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