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# The sharp-interface limit for the Navier–Stokes–Korteweg equations

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**Abstract**

We investigate the sharp-interface limit for the Navier–Stokes–Korteweg model, which is an extension of the compressible Navier–Stokes equations. By means of compactness arguments, we show that solutions of the Navier–Stokes– Korteweg equations converge to solutions of a physically meaningful free-boundary problem. Assuming that an associated energy functional converges in a suitable sense, we obtain the sharp-interface limit at the level of weak solutions.

## 1 Introduction

Models describing liquid-vapour flow are basically classified into two different types: sharp- and diffuse-interface models. They differ in how the interface dividing liquid from vapour is represented. In diffuse-interface models, an additional order parameter (here, the density) is introduced, such that the interface is described in a different manner. The "sharp interface" is replaced by an interfacial layer of positive thickness $\varepsilon$, where the order parameter varies rapidly but smoothly between two values distinguishing the liquid and the vapour phase. The sharp-interface limit encodes the behaviour of diffuse-interface models and their corresponding solutions, as $\varepsilon$ tends to zero.

We shall investigate the sharp-interface limit of the following "phase-field-like scaling" of the Navier–Stokes–Korteweg equations [13, 15]. In a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with $C^2$-boundary $\partial\Omega$ and outer unit normal $\nu$, we consider, for the unknowns density $\rho_\varepsilon$ and velocity $v_\varepsilon$, the partial differential equations

$$\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon v_\varepsilon) = 0, \tag{1}$$

$$\partial_t(\rho_\varepsilon v_\varepsilon) + \operatorname{div}(\rho_\varepsilon v_\varepsilon \otimes v_\varepsilon) + \tfrac{1}{\varepsilon}\nabla p(\rho_\varepsilon) = 2\operatorname{div}(\mu(\rho_\varepsilon) D v_\varepsilon) + \varepsilon \rho_\varepsilon \nabla\Delta\rho_\varepsilon, \tag{2}$$

depending on a parameter $\varepsilon \in (0,1)$, in the space-time cylinder $\Omega \times (0,T)$ with $T \in (0,\infty)$. In (2), $D$ stands for the symmetric part of the gradient. We close the system by adding the boundary and initial conditions

$$\nabla\rho_\varepsilon \cdot \nu = 0 \qquad \text{on } \partial\Omega \times [0,T), \tag{3}$$

$$v_\varepsilon = 0 \qquad \text{on } \partial\Omega \times [0,T), \tag{4}$$

$$\rho_\varepsilon(\cdot,0) = \rho_\varepsilon^{(i)} \qquad \text{in } \Omega, \tag{5}$$

$$v_\varepsilon(\cdot,0) = v_\varepsilon^{(i)} \qquad \text{in } \Omega. \tag{6}$$

The non-monotone pressure function $p = p(\rho)$ is given by the relation $p'(\rho) = \rho \widetilde{W}''(\rho) = \rho W''(\rho)$, where $W \in C^2([0,\infty))$ is a non-negative double-well potential, such that $W(z) = 0$ if and only if $z \in \{\beta_1, \beta_2\}$, and

$$W''(z) \geq C_1 |z-a|^{p_*-2} \text{ for all } z \in [0,\infty) \text{ with } |z-a| \geq b - C_2, \tag{7}$$

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where $a = \frac{\beta_1+\beta_2}{2}$ and $b = \frac{\beta_2-\beta_1}{2}$, for some constants $C_1 > 0$, $C_2 \in (0,b)$ and $p_* > 2$. As a direct consequence of (7), there exist constants $C_1, C_2 > 0$, such that

$$W(z) \geq C_1 |z-a|^{p_*} - C_2 \text{ and } (|z-a|-b)^2 \leq C_1 W(z) \text{ for all } z \in [0,\infty). \tag{8}$$

The viscosity function $\mu\colon [0,\infty) \to [c_\mu, C_\mu]$, $0 < c_\mu \leq C_\mu$, is Lipschitz continuous. For well-posedness results for (1)–(6) and related models see [3, 4, 5, 7, 11, 12, 16, 17].

Our work summarizes the results of [8, Chapters 3 and 5], where detailed proofs are given, and extends [13], where the static case of (1)–(6) is treated, to the dynamic case. We study (weak) solutions to (1)–(6) as $\varepsilon$ tends to zero, and seek to extract subsequences of $(\rho_\varepsilon, v_\varepsilon)_{\varepsilon\in(0,1)}$ converging to solutions $(\rho_0, v_0)$ of an appropriate sharp-interface model. We prove that $(\rho_0, v_0)$ is a (weak) solution of the two-phase Navier–Stokes equations with surface tension [14]: the free-boundary problem describing the motion of the vapour phase, of constant density $\beta_1$, and the liquid phase, of constant density $\beta_2$, of an isothermal, viscous, incompressible Newtonian fluid. For each time $t \in [0,T]$, a hypersurface $\Gamma(t)$ separates $\Omega$ into two disjoint open subsets $\Omega^-(t)$ and $\Omega^+(t)$ of $\Omega$, i.e., we have $\Omega = \Omega^-(t) \cup \Gamma(t) \cup \Omega^+(t)$ and $\Gamma(t) = \partial\Omega^-(t) \cap \Omega$. The unknowns are the free boundary $\Gamma(t)$, the velocity field $v(\cdot,t)\colon \Omega\setminus\Gamma(t) \to \mathbb{R}^n$ and the pressure function $p(\cdot,t)\colon \Omega\setminus\Gamma(t) \to \mathbb{R}$. The sharp-interface model then reads as

$$\beta_1\partial_t v + \beta_1(v\cdot\nabla)v - \mu(\beta_1)\Delta v + \nabla p = 0 \qquad \text{in } \Omega^-(t),\ t\in[0,T], \tag{9}$$

$$\beta_2\partial_t v + \beta_2(v\cdot\nabla)v - \mu(\beta_2)\Delta v + \nabla p = 0 \qquad \text{in } \Omega^+(t),\ t\in[0,T], \tag{10}$$

$$\operatorname{div}(v) = 0 \qquad \text{in } \Omega\setminus\Gamma(t),\ t\in[0,T], \tag{11}$$

$$[v] = 0 \qquad \text{on } \Gamma(t),\ t\in[0,T], \tag{12}$$

$$V = v\cdot\nu^- \qquad \text{on } \Gamma(t),\ t\in[0,T], \tag{13}$$

$$[T]\,\nu^- = -2\sigma_{\mathrm{st}}\kappa\nu^- \qquad \text{on } \Gamma(t),\ t\in[0,T]. \tag{14}$$

The stress tensor $T$ is given by $T(v(t),p(t)) = 2\mu(\beta_1)Dv(t) - p(t)I$ in $\Omega^-(t)$ and by $2\mu(\beta_2)Dv(t) - p(t)I$ in $\Omega^+(t)$. For a given quantity $f$, $[f]$ denotes the jump across $\Gamma(t)$ in the direction of the exterior unit-normal field $\nu^-(\cdot,t)$ of $\partial\Omega^-(t)$ (and pointing into $\Omega^+(t)$); that is, $[f](x,t) = \lim_{\xi\searrow 0}\big(f(x+\xi\nu^-(x,t),t) - f(x-\xi\nu^-(x,t),t)\big)$ for $x\in\Gamma(t)$, and $V$ and $\kappa$ are the normal velocity and the mean curvature of $\Gamma$, both taken with respect to $\nu^-$. Moreover, $\sigma_{\mathrm{st}}$ is the surface-tension constant given by

$$\sigma_{\mathrm{st}} = \int_{\beta_1}^{\beta_2} \sqrt{\min\left\{\tfrac12 W(z), |z-a|^2 + b^2\right\}}\,\mathrm{d}z.$$

We close the system by the boundary and initial conditions

$$v(\cdot,t) = 0 \qquad \text{on } \partial\Omega,\ t\in[0,T], \tag{15}$$

$$\Omega^-(0) = \Omega^{-,(i)}, \tag{16}$$

$$v(\cdot,0) = v^{(i)} \qquad \text{in } \Omega, \tag{17}$$

where $v^{(i)}$ and $\Omega^{-,(i)}$ are prescribed data satisfying $\Omega^{-,(i)} \cap \partial\Omega = \emptyset$.

**Notation and Preliminaries** Let $U \subset \mathbb{R}^d$, $d\in\mathbb{N}$, be open or closed. The space of smooth and compactly supported functions in $U$ is denoted by $C_0^\infty(U)$, $C_{0,\sigma}^\infty(U)$ is the subspace of $C_0^\infty(U)$ of divergence-free functions and $C_0(U)$ is the closure of $C_0^\infty(U)$ with respect to the supremum norm. Moreover, for $Q\subset\mathbb{R}^d$, we define $C_{(0)}^\infty(Q) = \{u\colon Q\to\mathbb{R} : u = U|_Q,\ U\in C_0^\infty(\mathbb{R}^d),\ \operatorname{supp}(u)\subset Q\}$. For a measurable set $M\subset\mathbb{R}^d$ and $r\in[1,\infty]$, $L^r(M)$ and $L^r(M;X)$ denote the standard Lebesgue spaces of scalar and $X$-valued functions, respectively. $W^{k,r}(U)$ is the Sobolev space of order $k \in \mathbb{N}$ and integrability exponent $r$. By $W_0^{k,r}(U)$, we denote the closure of $C_0^\infty(U)$ in $W^{k,r}(U)$ and we set $H^k(U) = W^{k,2}(U)$ and $H_0^k(U) = W_0^{k,2}(U)$. Furthermore, $L^2_\sigma(U)$ and $H^1_{0,\sigma}(U)$ denote the closure of $C^\infty_{0,\sigma}(U)$ in $L^2(U)$ and $H^1(U)$, respectively. For a Banach space $Y$ and $\alpha \in (0,1)$, the space $C^0([0,T];Y)$ contains all continuous functions $f\colon [0,T] \to Y$ and the Hölder space $C^{0,\alpha}([0,T];Y)$ is the subspace of all $f \in C^0([0,T];Y)$ with finite norm

$$\|f\|_{C^{0,\alpha}([0,T];Y)} = \sup_{t\in[0,T]} \|f(t)\|_Y + \sup_{0\le t_1<t_2\le T} \frac{\|f(t_2)-f(t_1)\|_Y}{|t_2-t_1|^\alpha} < \infty.$$

For $N \in \mathbb{N}$ and a finite $\mathbb{R}^N$-valued Radon measure $\mu$ and a Borel set $E \subset U$, the total-variation measure of $E$ is defined by

$$|\mu|(E) = \sup \sum_{m=1}^\infty |\mu(E_m)|,$$

where the supremum is taken over all pairwise disjoint partitions $(E_m)_{m\in\mathbb{N}} \subset X$ of measurable sets $E_m$, $m \in \mathbb{N}$, such that $E = \bigcup_{m=1}^\infty E_m$. A function $u \in L^1(U)$ is said to be of bounded variation if its distributional gradient $\nabla u$ is a finite $\mathbb{R}^d$-valued Radon measure. The set of all functions of bounded variation is denoted by $BV(U)$, and the set $BV(U,M)$ contains all functions $u \in BV(U)$, such that $u \in M$ for a.e. $x \in U$. A measurable set $E \subset U$ has finite perimeter in $U$ if its characteristic function $\chi_E$ belongs to $BV(U)$. By the structure theorem of sets of finite perimeter, there holds $|\nabla\chi_E|(U) = \mathscr{H}^{d-1}(U \cap \partial^* E)$, where $\mathscr{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure and $\partial^* E$ is the so-called reduced boundary of $E$ and, for all $\psi \in C_0^\infty(U)^d$,

$$\int_E \operatorname{div}(\psi)\,\mathrm{d}x = \int_{\partial^* E} \psi \cdot \nu_E \,\mathrm{d}\mathscr{H}^{d-1}(x),$$

where $\nu_E(x) = -\lim_{\delta\searrow 0} \frac{\nabla\chi_E(B_\delta(x))}{|\nabla\chi_E|(B_\delta(x))}$ is the generalized outer unit normal; cf. e.g. [2, Theorem 3.36]. Note that, if $E$ has $C^1$-boundary, then $\partial^* E = \partial E$ and $\nu_E$ coincides with the outer unit normal.

**Organisation** In Section 2 we establish compactness properties for weak solutions of the Navier–Stokes–Korteweg equations. Finally, in Section 3, we obtain the main theorem: the sharp-interface limit for weak solutions of (1)–(6), assuming the convergence of an associated energy functional.

## 2 A Priori Estimates and Compactness Properties

For the prescribed data in (5) and (6), for every $\varepsilon \in (0,1)$, let $v_\varepsilon^{(i)} \in H_0^1(\Omega)^n$ and assume that $\rho_\varepsilon^{(i)} \in L^{p_*}(\Omega) \cap H^1(\Omega)$ satisfies $\rho_\varepsilon^{(i)} \ge 0$ a.e. in $\Omega$ and

$$\int_\Omega \rho_\varepsilon^{(i)}\,\mathrm{d}x = m,$$

for some $m \in (\beta_1|\Omega|, \beta_2|\Omega|)$, Moreover, we assume the following asymptotic behaviour. Here and subsequently, all limits are taken as $\varepsilon \to 0$, and as $j \to \infty$ along subsequences $(\varepsilon_j)_{j\in\mathbb{N}}$. We suppose that $v_\varepsilon^{(i)} \to v_0^{(i)}$ in $H^1(\Omega)^n$ for some $v_0^{(i)} \in H^1_{0,\sigma}(\Omega)$ and that $\rho_\varepsilon^{(i)} \to \rho_0^{(i)}$ in $L^{p_*}(\Omega)$ for some $\rho_0^{(i)} \in BV(\Omega, \{\beta_1,\beta_2\})$, such that, for a set $\Omega_0^{(i)} \subset\subset \Omega$ of finite perimeter with characteristic function $\chi_0^{(i)} = \chi_{\Omega_0^{(i)}}$,

$$\rho_0^{(i)} = (\beta_1-\beta_2)\chi_0^{(i)} + \beta_2 = (\beta_1-\beta_2)\chi_{\Omega_0^{(i)}} + \beta_2 \text{ a.e. in } \Omega. \tag{18}$$

Finally, letting $\Gamma_0^{(i)} = \partial^*(\Omega_0^{(i)})$, we suppose that

$$E_\varepsilon^{\mathrm{tot},(i)} = \int_\Omega \frac{1}{\varepsilon} W(\rho_\varepsilon^{(i)}) + \frac{1}{2}\varepsilon \left|\nabla \rho_\varepsilon^{(i)}\right|^2 + \frac{1}{2}\rho_\varepsilon^{(i)} \left|v_\varepsilon^{(i)}\right|^2 \mathrm{d}x$$

converges to

$$2\sigma_{\mathrm{st}} \mathscr{H}^{n-1}(\Gamma_0^{(i)}) + \frac{1}{2}\int_\Omega \rho_0^{(i)} \left|v_0^{(i)}\right|^2 \mathrm{d}x.$$

**Weak Formulation** For $t \in [0,T)$, consider the corresponding energy functionals

$$E_\varepsilon(t) = \int_\Omega \tfrac{1}{\varepsilon} W(\rho_\varepsilon) + \tfrac{\varepsilon}{2}\left|\nabla \rho_\varepsilon\right|^2 \mathrm{d}x \text{ and } E_\varepsilon^{\mathrm{tot}}(t) = E_\varepsilon(t) + \int_\Omega \tfrac{1}{2}\rho_\varepsilon(t)\left|v_\varepsilon(t)\right|^2 \mathrm{d}x.$$

For sufficiently smooth solutions of (1)–(6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\varepsilon^{\mathrm{tot}}(t) = -2\int_\Omega \mu(\rho_\varepsilon(t))\left|Dv_\varepsilon(t)\right|^2 \mathrm{d}x$$

and, in particular, $E_\varepsilon^{\mathrm{tot}}$ is non-increasing; see [8, Theorem 3.2.3] for details. This motivates the following weak formulation [8, Definition 3.2.5].

**Definiton 1.** A pair

$$(\rho_\varepsilon, v_\varepsilon) \in \left(L^\infty(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^{p_*}(\Omega))\right) \times L^2(0,T;H_0^1(\Omega)^n)$$

with $\rho_\varepsilon \geq 0$ a.e. in $\Omega \times (0,T)$ is called a weak solution to (1)–(6) if the following conditions are satisfied.

1 For all $\varphi \in C_{(0)}^\infty(\overline{\Omega} \times [0,T))$, there holds

$$\int_0^T \int_\Omega \rho_\varepsilon \partial_t \varphi + \rho_\varepsilon v_\varepsilon \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \rho_\varepsilon^{(i)} \varphi(0)\,\mathrm{d}x = 0. \tag{19}$$

2 There holds

$$\begin{aligned} &\int_0^T \int_\Omega \rho_\varepsilon v_\varepsilon \cdot \partial_t \psi + \rho_\varepsilon v_\varepsilon \otimes v_\varepsilon : \nabla \psi + \tfrac{1}{\varepsilon} p(\rho_\varepsilon)\,\mathrm{div}(\psi) \\ &= \int_0^T \int_\Omega 2\mu(\rho_\varepsilon) Dv_\varepsilon : D\psi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_\varepsilon^{(i)} v_\varepsilon^{(i)} \cdot \psi(0)\,\mathrm{d}x \\ &\quad - \varepsilon \int_0^T \int_\Omega \nabla \rho_\varepsilon \otimes \nabla \rho_\varepsilon : \nabla \psi + \tfrac{1}{2}\left|\nabla \rho_\varepsilon\right|^2 \mathrm{div}(\psi) + \rho_\varepsilon \nabla \rho_\varepsilon \cdot \nabla \mathrm{div}(\psi)\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{20}$$

for all $\psi \in C_{(0)}^\infty(\Omega \times [0,T))^n$.

3 For a.e. $\tau_1 \in [0,T)$, including $\tau_1 = 0$, there holds, for all $\tau_2 \in [\tau_1, T)$,

$$E_\varepsilon^{\mathrm{tot}}(\tau_2) + 2\int_{\tau_1}^{\tau_2} \int_\Omega \mu(\rho_\varepsilon)\left|Dv_\varepsilon\right|^2 \mathrm{d}x\,\mathrm{d}t \leq E_\varepsilon^{\mathrm{tot}}(\tau_1). \tag{21}$$

We assume that solutions in the sense of the above definition exist. However, to the best of our knowledge, there is no global-existence result known. Kotschote [16, 17] proved the (short-time) existence of strong solutions. This, in particular, guarantees the existence of solutions in sense Definition 2 for short times. Note that, the interval of existence may depend on $\varepsilon$, in general. Haspot [11] investigated the existence of global weak solutions in the case of a (non-physical) monotone pressure function.

**A Priori Estimates** For weak solutions to (1)–(6), we shall prove a priori estimates and establish appropriate compactness properties, adapting arguments due to Chen [6]. As a consequence of (8) and (21), we get

$$\|\rho_\varepsilon\|_{L^\infty(0,T;L^{p_*}(\Omega))} + \varepsilon^{-1/2}\| \, |\rho_\varepsilon - a| - b\|_{L^\infty(0,T;L^2(\Omega))} + \|v_\varepsilon\|_{L^2(0,T;H^1(\Omega)^n)} \leq C \tag{22}$$

for some constant $C > 0$. We introduce the transformation

$$\Phi(s) = \int_a^s \sqrt{\min\left\{\tfrac{1}{2}W(z), |z-a|^2 + b^2\right\}} \,\mathrm{d}z \text{ for } s \in [0,\infty).$$

By standard arguments [6, p. 276], $(r_\varepsilon)_{\varepsilon\in(0,1)}$, defined by $r_\varepsilon = \Phi \circ \rho_\varepsilon$, is bounded in $L^\infty(0,T;BV(\Omega))$. For a standard mollifying kernel $\Theta$ and sufficiently small $\eta > 0$, let

$$\rho_{\varepsilon,\eta}(x,t) = \int_{B_1(0)} \Theta(y)\rho_\varepsilon(x-\eta y,t)\,\mathrm{d}y \text{ for } (x,t) \in \Omega \times [0,T],$$

where $\rho_\varepsilon$ is extended to a small neighbourhood of $\Omega$ as in [6, Proof of Lemma 3.2]. Proceeding analogously to [6, equations (3.2)–(3.4)], there exists a constant $C > 0$, such that, for $t \in [0,T]$ and sufficiently small $\eta > 0$,

$$\|\rho_{\varepsilon,\eta}(t) - \rho_\varepsilon(t)\|_{L^2(\Omega)} \leq C\sqrt{\eta} \text{ and } \|\nabla\rho_{\varepsilon,\eta}(t)\|_{L^3(\Omega)^n} \leq C\eta^{-(\frac{2}{3}n+1)}. \tag{23}$$

Then, $(\rho_\varepsilon)_{\varepsilon\in(0,1)} \subset C^{0,\frac{1}{28}}([0,T];L^2(\Omega))$ and $(r_\varepsilon)_{\varepsilon\in(0,1)} \subset C^{0,\frac{1}{28}}([0,T];L^1(\Omega))$ are bounded, by the arguments of Chen [6, Lemma 3.2]; see [8, Theorem 3.3.11] for details. For convenience of the reader, we briefly ensure the existence of a constant $C > 0$, such that, for all $t_1,t_2 \in [0,T]$ with $t_1 < t_2$ and $|t_2 - t_1|$ sufficiently small,

$$\int_\Omega |\rho_\varepsilon(t_2) - \rho_\varepsilon(t_1)|^2 \,\mathrm{d}x \leq C\,|t_2 - t_1|^{\frac{1}{14}}. \tag{24}$$

Due to (22) and $H^1(\Omega) \hookrightarrow L^6(\Omega)$, $(\rho_\varepsilon)_{\varepsilon\in(0,1)}$ and $(v_\varepsilon)_{\varepsilon\in(0,1)}$ are bounded in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;L^6(\Omega)^n)$, respectively. From (19), we obtain

$$\rho_\varepsilon(t_2) - \rho_\varepsilon(t_1) = \int_{t_1}^{t_2} \partial_t\rho_\varepsilon(t)\,\mathrm{d}t = -\int_{t_1}^{t_2} \operatorname{div}(\rho_\varepsilon v_\varepsilon)(t)\,\mathrm{d}t \text{ in } W^{1,3}(\Omega)^*.$$

Hence, recalling that $v_\varepsilon \in L^2(0,T;H_0^1(\Omega)^n)$ and using (23), we infer

$$\begin{aligned}
&\int_\Omega (\rho_\varepsilon(t_2) - \rho_\varepsilon(t_1))(\rho_{\varepsilon,\eta}(t_2) - \rho_{\varepsilon,\eta}(t_1))\,\mathrm{d}x \\
&= -\int_{t_1}^{t_2} \left\langle \operatorname{div}(\rho_\varepsilon v_\varepsilon)(t), \rho_{\varepsilon,\eta}(t_2) - \rho_{\varepsilon,\eta}(t_1)\right\rangle_{W^{1,3}(\Omega)} \mathrm{d}t \\
&= \int_{t_1}^{t_2}\int_\Omega (\rho_\varepsilon v_\varepsilon)(t)\cdot(\nabla\rho_{\varepsilon,\eta}(t_2) - \nabla\rho_{\varepsilon,\eta}(t_1))\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \|\rho_\varepsilon v_\varepsilon\|_{L^2(t_1,t_2;L^{\frac{3}{2}}(\Omega)^n)} \|\nabla\rho_{\varepsilon,\eta}(t_2) - \nabla\rho_{\varepsilon,\eta}(t_1)\|_{L^2(t_1,t_2;L^3(\Omega)^n)} \\
&\leq 2\|\rho_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}\|v_\varepsilon\|_{L^2(0,T;L^6(\Omega)^n)} \sup_{t\in[0,T]} \|\nabla\rho_{\varepsilon,\eta}(t)\|_{L^3(\Omega)^n}\,|t_2 - t_1|^{\frac{1}{2}} \\
&\leq C\eta^{-(\frac{2}{3}n+1)}\,|t_2 - t_1|^{\frac{1}{2}}.
\end{aligned}$$

Using $\rho_\varepsilon(t_2)-\rho_\varepsilon(t_1)=\rho_\varepsilon(t_2)\mp\rho_{\varepsilon,\eta}(t_2)\mp\rho_{\varepsilon,\eta}(t_1)-\rho_\varepsilon(t_1)$ and Hölder's inequality, in view of (23), there exists a constant $C>0$, such that, for sufficiently small $\eta>0$,

$$\int_\Omega |\rho_\varepsilon(t_2)-\rho_\varepsilon(t_1)|^2\,\mathrm{d}x \le C\Big(\sqrt{\eta}+\eta^{-(\frac{2}{3}n+1)}\,|t_2-t_1|^{\frac{1}{2}}\Big).$$

Since $n\in\{2,3\}$, the choice $\eta=|t_2-t_1|^{\frac{1}{7}}$ implies (24) for $|t_2-t_1|$ sufficiently small.

**Compactness** Throughout this paper, we will not relabel subsequences. As a direct consequence of (22), there exists $v_0\in L^2(0,T;H^1_0(\Omega)^n)$, such that, after passing to a subsequence, $v_\varepsilon \rightharpoonup v_0$ in $L^2(0,T;H^1(\Omega)^n)\hookrightarrow L^2(0,T;L^6(\Omega)^n)$.

We will use the following adaptation of Simon [19, Theorem 3]. Let $0<\alpha<\beta$ and let $X,Y$ be Banach spaces, such that $X\hookrightarrow\hookrightarrow Y$. Let $(f_k)_{k\in\mathbb{N}}\subset C^{0,\beta}([0,T];Y)$ be bounded and let $f\in C^0([0,T];Y)$. If $f_k\to f$ in $C^0([0,T];Y)$ for $k\to\infty$, then $f\in C^{0,\beta}([0,T];Y)$, and $f_k\to f$ in $C^{0,\alpha}([0,T];Y)$ as $k\to\infty$. Moreover, $L^\infty(0,T;X)\cap C^{0,\beta}([0,T];Y)\hookrightarrow\hookrightarrow C^{0,\alpha}([0,T];Y)$. By applying Lemma 2 to $(r_\varepsilon)_{\varepsilon\in(0,1)}$, there exist a subsequence $(r_{\varepsilon_j})_{j\in\mathbb{N}}$ and $r_0\in L^\infty(0,T;BV(\Omega))\cap C^{0,\frac{1}{28}}([0,T];L^1(\Omega))$, such that, there holds

$$r_{\varepsilon_j}\to r_0 \text{ in } C^{0,\frac{1}{29}}([0,T];L^1(\Omega)) \tag{25}$$

and $|\nabla r_0(t)|(\Omega)\le \liminf_{j\to\infty}\left|\nabla r_{\varepsilon_j}(t)\right|(\Omega)$ for every $t\in[0,T]$.

Using the properties of the transformation $\Phi$ and Lemma 2, (25) implies that $\rho_0=\Phi^{-1}\circ r_0\in C^{0,\frac{1}{28}}([0,T];L^2(\Omega))$ and $\rho_\varepsilon\to\rho_0$ in $C^{0,\frac{1}{29}}([0,T];L^2(\Omega))$. In particular, for any $t\in[0,T]$, $\rho_\varepsilon(t)\to\rho_0(t)$ in $L^2(\Omega)$, hence, (22) implies $\rho_\varepsilon(t)\rightharpoonup\rho_0(t)$ in $L^{p_*}(\Omega)$. Finally, by interpolation, for any $q\in[1,p_*)$, it follows $\rho_\varepsilon(t)\rightharpoonup\rho_0(t)$ in $L^{p_*}(\Omega)$. Note that, by the preceding results, for every $t\in[0,T]$, it follows that $\rho_0(t)\in BV(\Omega,\{\beta_1,\beta_2\})$ and

$$\rho_0(t)=(\beta_1-\beta_2)\chi_0+\beta_2=(\beta_1-\beta_2)\chi_{\Omega^-(t)}+\beta_2 \text{ a.e. in } \Omega, \tag{26}$$

where $\chi_0=\frac{\rho_0-\beta_2}{\beta_1-\beta_2}$ and $\Omega^-(t)$ is the set of finite perimeter in $\Omega$ given by

$$\Omega^-(t)=\left\{x\in\Omega : \lim_{\delta\to 0}\frac{1}{|B_\delta(x)|}\int_{B_\delta(x)}\chi_0(y,t)\,\mathrm{d}y=1\right\}. \tag{27}$$

We call $\Omega^-(\cdot)$ the measure-theoretic representative set of $\rho_0$. In this way, $\rho_0(t)$ induces the disjoint partition $\Omega=\Omega^-(t)\cup\Gamma(t)\cup\Omega^+(t)$, where the (sharp) interface $\Gamma(t)$ and $\Omega^+(t)$ are, respectively, defined by

$$\Gamma(t)=\partial^*\Omega^-(t)\cap\Omega \text{ and } \Omega^+(t)=\Omega\setminus(\Omega^-(t)\cup\Gamma(t)).$$

Note that, in view of the generalised Gauß–Green theorem, the generalised measure-theoretic outer normal $\nu^-(t)$ exists on $\Gamma(t)$.

## 3 The Sharp-Interface Limit and Main Theorem

We investigate the sharp-interface limit for weak solutions of (1)–(6) along suitable subsequences under the additional assumptions on the energy functionals $E_\varepsilon$ and $E_\varepsilon^{\mathrm{tot}}$ given below in (28) and (29). To simplify notation, we assume that any convergence property of $(\rho_\varepsilon,v_\varepsilon)_{\varepsilon\in(0,1)}$ holds true for the entire sequence and not only for an appropriate subsequence.

**Assumptions** To identify the limit of $E_\varepsilon^{\mathrm{tot}}$, suppose that, for any $t \in [0,T]$, $\Gamma(t)$ is compactly contained in $\Omega$, and that, for every $\varphi \in L^1(0,T;C_0(\overline{\Omega}))$,

$$\int_0^T \int_\Omega \left(\tfrac{1}{\varepsilon} W(\rho_\varepsilon) + \tfrac{\varepsilon}{2} |\nabla \rho_\varepsilon|^2\right) \varphi \,\mathrm{d}x\,\mathrm{d}t \to 2\sigma_{\mathrm{st}} \int_0^T \int_{\Gamma(t)} \varphi \,\mathrm{d}\mathscr{H}^{n-1}(x)\,\mathrm{d}t. \tag{28}$$

Additionally, assume the following asymptotic behaviour of the kinetic part of $E_\varepsilon^{\mathrm{tot}}$:

$$\sqrt{\rho_\varepsilon} v_\varepsilon \to \sqrt{\rho_0} v_0 \text{ in } L^2(0,T;L^2(\Omega)^n). \tag{29}$$

Using the reasoning of [18, Lemmata 1 and 2], (28) implies the so-called equipartition-of-energy property

$$\int_0^T \int_\Omega \left| \tfrac{1}{\varepsilon} W(\rho_\varepsilon) - \tfrac{1}{2} \varepsilon |\nabla \rho_\varepsilon|^2 \right| \mathrm{d}x\,\mathrm{d}t \to 0, \tag{30}$$

and, moreover, for any $\psi \in C_0^\infty([0,T);C_{0,\sigma}^\infty(\Omega))$, there holds

$$\varepsilon \int_0^T \int_\Omega \nabla \rho_\varepsilon \otimes \nabla \rho_\varepsilon : \nabla \psi \,\mathrm{d}x\,\mathrm{d}t \to 2\sigma_{\mathrm{st}} \int_0^T \int_{\Gamma(t)} \nu^- \otimes \nu^- : \nabla \psi \,\mathrm{d}\mathscr{H}^{n-1}(x)\,\mathrm{d}t. \tag{31}$$

After passing to a suitable subsequence, in view of (21) and (29), we obtain

$$\int_\Omega \rho_{\varepsilon_{j_m}} \left| v_{\varepsilon_{j_m}} \right|^2 \mathrm{d}x \overset{*}{\rightharpoonup} \int_\Omega \rho_0 |v_0|^2 \,\mathrm{d}x \text{ in } L^\infty(0,T) \cong L^1(0,T)^*. \tag{32}$$

**Weak formulation** First, we introduce a weak formulation of (9)–(17). For its derivation and justification, we refer to [8, Chapter 4]. For the prescribed data in (16) and (17), we assume that $v^{(i)} \in H_{0,\sigma}^1(\Omega)$ and $\Omega^{-,(i)} \subset\subset \Omega$ with corresponding characteristic function $\chi^{(i)} = \chi_{\Omega^-(0)}$. Moreover, let $\rho^{(i)} \in BV(\Omega,\{\beta_1,\beta_2\})$ given by $\rho^{(i)} = (\beta_1 - \beta_2)\chi^{(i)} + \beta_2$.

**Definition 2.** A pair

$$(\rho, v) \in L^\infty(0,T;BV(\Omega,\{\beta_1,\beta_2\})) \times \left(L^\infty(0,T;L_\sigma^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)^n)\right)$$

is called a weak solution of (9)–(17) if the following conditions are fulfilled.

1 The measure-theoretic representative set $\Omega^-(t)$ of $\rho(t)$, cf. (27), is compactly contained in $\Omega$; that is, for a.e. $t \in (0,T)$, there holds $\Omega^-(t) \subset\subset \Omega$.

2 For each $\psi \in C_0^\infty([0,T);C_{0,\sigma}^\infty(\Omega))$, there holds

$$\begin{aligned}
&\int_0^T \int_\Omega \rho v \cdot \partial_t \psi + \rho v \otimes v : \nabla \psi - 2\mu(\rho) Dv : D\psi \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_\Omega \rho^{(i)} v^{(i)} \cdot \psi(0)\,\mathrm{d}x - 2\sigma_{\mathrm{st}} \int_0^T \int_{\Gamma(t)} \nu^- \otimes \nu^- : \nabla \psi \,\mathrm{d}\mathscr{H}^{n-1}(x)\,\mathrm{d}t,
\end{aligned} \tag{33}$$

where $\Gamma(t) = \partial^*(\Omega^-(t))$ with generalised outer unit normal $\nu^-(t)$.

3 For a.e. $\tau_1 \in [0,T)$, including $\tau_1 = 0$, there holds, for all $\tau_2 \in [\tau_1, T)$,

$$\begin{aligned}
&2\sigma_{\mathrm{st}} \mathscr{H}^{n-1}(\Gamma(\tau_2)) + \tfrac{1}{2} \int_\Omega \rho(\tau_2) |v(\tau_2)|^2 \,\mathrm{d}x + 2 \int_{\tau_1}^{\tau_2} \int_\Omega \mu(\rho) |Dv|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\le 2\sigma_{\mathrm{st}} \mathscr{H}^{n-1}(\Gamma(\tau_1)) + \tfrac{1}{2} \int_\Omega \rho(\tau_1) |v(\tau_1)|^2 \,\mathrm{d}x.
\end{aligned} \tag{34}$$

4 Let $\chi = \frac{\rho-\beta_2}{\beta_1-\beta_2}$. For every $\varphi \in C^\infty_{(0)}(\overline{\Omega}\times[0,T))$, there holds

$$\int_0^T \int_\Omega \chi(\partial_t\varphi + v\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \chi^{(i)}(x)\varphi(0)\,\mathrm{d}x = 0. \tag{35}$$

**Energy Inequality** Next, we prove that, for all $\tau_1 \le \tau_2 < T$ and almost all $0 \le \tau_1 < T$, including $\tau_1 = 0$, there holds

$$\begin{aligned} &2\sigma_{\mathrm{st}}\mathscr{H}^{n-1}(\Gamma(\tau_2)) + \tfrac{1}{2}\int_\Omega \rho_0(\tau_2)\,|v_0(\tau_2)|^2\,\mathrm{d}x + 2\int_{\tau_1}^{\tau_2}\int_\Omega \mu(\rho_0)\,|Dv_0|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\le 2\sigma_{\mathrm{st}}\mathscr{H}^{n-1}(\Gamma(\tau_1)) + \tfrac{1}{2}\int_\Omega \rho_0(\tau_1)\,|v_0(\tau_1)|^2\,\mathrm{d}x. \end{aligned} \tag{36}$$

By (21) and [1, Lemma 4.3], for all $\tau \in W^{1,1}(0,T)$ with $\tau \ge 0$ and $\tau(T) = 0$, we obtain

$$E^{\mathrm{tot}}_\varepsilon(0)\tau(0) + \int_0^T E^{\mathrm{tot}}_\varepsilon(t)\tau'(t)\,\mathrm{d}t \ge 2\int_0^T\int_\Omega \mu(\rho_\varepsilon)\,|Dv_\varepsilon|^2\,\mathrm{d}x\,\tau(t)\,\mathrm{d}t. \tag{37}$$

Since $\mu$ is non-negative and Lipschitz continuous, $\rho_\varepsilon \to \rho_0$ in $L^\infty(0,T;L^2(\Omega))$, $Dv_\varepsilon \rightharpoonup Dv_0$ in $L^2(0,T;L^2(\Omega)^{n\times n})$ and (21) holds, we conclude that, after a possible passage to an appropriate subsequence, there holds

$$\sqrt{\mu(\rho_\varepsilon)}Dv_\varepsilon \rightharpoonup \sqrt{\mu(\rho_0)}Dv_0 \text{ in } L^2(0,T;L^2(\Omega)^{n\times n}).$$

Using $W^{1,1}(0,T) \hookrightarrow L^\infty(0,T)$, $\tau \ge 0$ and the lower semi-continuity of the $L^2$-norm with respect to weak convergence, we obtain

$$\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega \mu(\rho_\varepsilon(t))\,|Dv_\varepsilon(t)|^2\,\mathrm{d}x\,\tau(t)\,\mathrm{d}t \ge \int_0^T\int_\Omega \mu(\rho_0(t))\,|Dv_0(t)|^2\,\mathrm{d}x\,\tau(t)\,\mathrm{d}t.$$

By the convergence of $E^{\mathrm{tot},(i)}_\varepsilon$, (28) and (32), taking (37) to the limit leads to

$$\begin{aligned} &\left(2\sigma_{\mathrm{st}}\mathscr{H}^{n-1}(\Gamma_0^{(i)}) + \tfrac{1}{2}\int_\Omega \rho_0^{(i)}\left|v_0^{(i)}\right|^2\,\mathrm{d}x\right)\tau(0) \\ &+ \int_0^T \left(2\sigma_{\mathrm{st}}\mathscr{H}^{n-1}(\Gamma(t)) + \tfrac{1}{2}\int_\Omega \rho_0(t)\,|v_0(t)|^2\,\mathrm{d}x\right)\tau'(t)\,\mathrm{d}t \\ \ge\, &2\int_0^T\int_\Omega \mu(\rho_0(t))\,|Dv_0(t)|^2\,\mathrm{d}x\,\tau(t)\,\mathrm{d}t. \end{aligned}$$

Finally, another application of [1, Lemma 4.3] gives (36).

**Regularity of Limiting Velocity and Transport Equation** Taking (19) to the limit, and using the energy estimate (36), we may conclude that the pair $(\rho_0, v_0)$ has the desired regularity and that $\chi_0$ satisfies (35). As $\rho_0(t) \ge \beta_1 > 0$ a.e. in $\Omega$, for $t \in (0,T)$, the energy estimate (36) yields

$$2\sigma_{\mathrm{st}}\mathscr{H}^{n-1}(\Gamma(t)) + \tfrac{1}{2}\beta_1\int_\Omega |v_0(t)|^2\,\mathrm{d}x \le 2\sigma_{\mathrm{st}}\mathscr{H}^{n-1}(\Gamma_0^{(i)}) + \tfrac{1}{2}\int_\Omega \rho_0^{(i)}\left|v_0^{(i)}\right|^2\,\mathrm{d}x.$$

Recalling that $|\nabla\chi_0(t)|(\Omega) = \mathscr{H}^{n-1}(\Gamma(t))$, implies that $\chi_0 \in L^\infty(0,T;BV(\Omega))$ and $v_0 \in L^\infty(0,T;L^2(\Omega)^n)$. Taking (19) to the limit leads to

$$\int_0^T\int_\Omega \rho_0\partial_t\varphi + \rho_0 v_0\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \rho_0^{(i)}\varphi(0)\,\mathrm{d}x = 0 \tag{38}$$

for any $\varphi \in C^\infty_{(0)}(\overline{\Omega}\times[0,T))$. By [10, Theorem 10.29], $\rho_0$ is a renormalized solution in the sense of DiPerna and Lions [9], which means that there holds

$$\int_0^T\int_\Omega b(\rho_0)\partial_t\varphi + b(\rho_0)v_0\cdot\nabla\varphi - (\rho_0 b'(\rho_0) - b(\rho_0))\,\mathrm{div}(v_0)\varphi\,\mathrm{d}x\,\mathrm{d}t = 0$$

for any $b \in C^1([0,\infty))\cap W^{1,\infty}(0,\infty)$ and any $\varphi \in C_0^\infty(\Omega\times(0,T))$. Choosing $b$ such that $b(\beta_1) = b(\beta_2) = 0$, $b'(\beta_1) = \frac{1}{\beta_1}$ and $b'(\beta_2) = \frac{1}{\beta_2}$, and recalling that $\rho_0(t)\in\{\beta_1,\beta_2\}$ a.e. in $\Omega$, yields

$$\int_0^T\int_\Omega v_0\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_\Omega \mathrm{div}(v_0)\varphi\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{39}$$

Hence $v_0 \in L^\infty(0,T;L^2_\sigma(\Omega))$. Plugging (18) and (26) into (38) implies

$$\begin{aligned}&(\beta_1-\beta_2)\left(\int_0^T\int_\Omega \chi_0(\partial_t\varphi + v_0\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \chi_0^{(i)}\varphi(0)\,\mathrm{d}x\right)\\ &= -\beta_2\int_0^T\int_\Omega v_0\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{40}$$

for any $\varphi \in C^\infty_{(0)}(\overline{\Omega}\times[0,T))$. Due to (39), we finally infer that $\chi_0$ satisfies (35).

**Variational Formulation** For any $\psi \in C_0^\infty([0,T);C^\infty_{0,\sigma}(\Omega))$, by (20), there holds

$$\begin{aligned}&\int_0^T\int_\Omega \rho_\varepsilon v_\varepsilon\partial_t\psi + \rho_\varepsilon v_\varepsilon\otimes v_\varepsilon : \nabla\psi - 2\mu(\rho_\varepsilon)Dv_\varepsilon : D\psi\,\mathrm{d}x\,\mathrm{d}t\\ &= -\int_\Omega \rho_\varepsilon^{(i)}v_\varepsilon^{(i)}\cdot\psi(0)\,\mathrm{d}x - \varepsilon\int_0^T\int_\Omega \nabla\rho_\varepsilon\otimes\nabla\rho_\varepsilon : \nabla\psi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{41}$$

Finally, we use the convergence properties established in Section 2 to take (41) to the limit. As $\mu$ is Lipschitz continuous, $\mu(\rho_\varepsilon)$ inherits the convergence properties of $\rho_\varepsilon$. Hence, using (29) and (31), we conclude that $(\rho_0, v_0)$ satisfies (33).

**Main theorem** We perform the sharp-interface limit and gather together the results of Sections 2 and 3 in the following theorem.

**Theorem 1.** *Let* (28) *and* (29) *hold true. Then there exist a subsequence* $(\rho_{\varepsilon_j}, v_{\varepsilon_j})_{j\in\mathbb{N}}$ *of* $(\rho_\varepsilon, v_\varepsilon)_{\varepsilon\in(0,1)}$ *and a pair* $(\rho_0, v_0)$ *with the following properties.*

*1* $\rho_0 \in C^{0,\frac{1}{28}}([0,T];L^2(\Omega))\cap L^\infty(0,T;BV(\Omega,\{\beta_1,\beta_2\}))$.

*2* $v_0 \in L^2(0,T;H_0^1(\Omega)^n)\cap L^\infty(0,T;L^2_\sigma(\Omega))$.

*3* $(\rho_0, v_0)$ *is a weak solution of* (9)–(17) *in the sense of Definition 3.*

*4 For any* $t\in[0,T]$*, as* $j\to\infty$*, there holds*

- *a.* $\rho_{\varepsilon_j}\to\rho_0$ *in* $C^{0,\frac{1}{29}}([0,T];L^2(\Omega))$,
- *b.* $\rho_{\varepsilon_j}(t)\to\rho_0(t)$ *in* $L^q(\Omega)$ *for any* $q\in[1,p_*)$,
- *c.* $\rho_{\varepsilon_j}(t)\rightharpoonup\rho_0(t)$ *in* $L^{p_*}(\Omega)$,
- *d.* $v_{\varepsilon_j}\rightharpoonup v_0$ *in* $L^2(0,T;H^1(\Omega)^n)$.

I. durch anderen Dateinamen ersetzen

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