# Smooth rough paths, their geometry and algebraic renormalization 

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#### Abstract

We introduce the class of "smooth rough paths" and study their main properties. Working in a smooth setting allows us to discard sewing arguments and focus on algebraic and geometric aspects. Specifically, a Maurer-Cartan perspective is the key to a purely algebraic form of Lyons extension theorem, the renormalization of rough paths in the spirit of [Bruned, Chevyrev, Friz, Preiß, A rough path perspective on renormalization, J. Funct. Anal. 277(11), 2019] as well as a related notion of "sum of rough paths". We first develop our ideas in a geometric rough path setting, as this best resonates with recent works on signature varieties, as well the renormalization of geometric rough paths. We then explore extensions to the quasi-geometric and the more general Hopf algebraic setting.


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## 1 Introduction

Recent years led to a remarkable convergence of different streams of mathematics. At the center of it is the notion of path $X:[0, T] \rightarrow \mathbb{R}^{d}$ and the "stack" of its iterated integrals

$$
\mathbf{X}_{T}:=\operatorname{Sig}\left(\left.X\right|_{[0, T]}\right):=\sum_{n \geq 0} \int_{0 \leq u_{1} \leq \ldots \leq u_{n} \leq T} \ldots \int_{u_{1}} d X_{u_{1}} \otimes \ldots \otimes d X_{u_{n}}
$$

commonly called the signature of $X$ over $[0, T]$, which takes values in $T\left(\left(\mathbb{R}^{d}\right)\right)$, the space of tensor series over $\mathbb{R}^{d}$. It is easy to see that the signature of a path segment actually takes its values in a very special curved subspace of the tensor (series) algebra, $G\left(\mathbb{R}^{d}\right) \subset$ $T\left(\left(\mathbb{R}^{d}\right)\right)$, with a natural group structure. This construction, which originates in Chen's fundamental work [Che54] is central to the theory of rough paths and stochastic analysis Lyo98, LCL07, FV10. Specifically, the path $t \mapsto \mathbf{X}_{t}:=\operatorname{Sig}\left(\left.X\right|_{[0, t]}\right)$ is the canonical rough path lift of $X$, for any sufficiently smooth path $X$ to make the signature well-defined. These ideas have proven useful in a remarkable variety of fields, stretching from machine learning CK16] to algebraic geometry and renormalization theory. More specifically, we mention:

Signature varieties. In AFS19, Améndola, Sturmfels and one of us study the geometry of signatures tensors. The signatures of a given class of smooth paths parametrize an algebraic variety inside the space of tensors, derived from the free nilpotent Lie group, with surprising analogies with the Veronese variety from algebraic geometry. These signature varieties provide both new tools to investigate paths and new challenging questions about their behaviour. In [AFS19] piecewise linear paths and polynomial paths are investigated. In a later work by Galuppi Gal19], and in the terminology of this paper, the role of classical smooth paths have been replaced by certain smooth rough paths, see Definition 2.1. Related recent works on signature varieties include CGMe20, PSS19.

Rough paths and renormalization. Rough paths were famously used to solve the singular KPZ stochastic partial differential equation Hai13 and subsequently led to the theory of regularity structures for general singular SPDEs Hai14], with precise correspondences to rough paths highlighted e.g. in [FH20, Sec. 13.2.2]. The central topic of renormalization of singular SPDEs [BHZ19] was revisited from a rough path perspective in BCFP19, which notably introduced pre-Lie algebras, and further inspired progress in the field [BCE20], see also Otto et al. [LOT21].

We cannot possibly expose in full the subtle intertwining of probabilistic, analytic, geometric and algebraic techniques of the above works, but still, sketch the general idea in a simple setting. If $X$ models a realization of noise, such as Brownian sample paths, then the very notion of Stieltjes integration against $d X_{u}$ is ill-defined (with probability one, such paths are not of locally bounded variation). The stochastic analysis provides probabilistic solutions: Stratonovich integration amounts to work with mollified $X$, followed by taken limits in probability, whereas Itô integration respects the martingale structure of such processes. Rough path theory (later: regularity structures) understands that these different calculi can be hard-coded imposing the first $N$-iterated integrals of $X$. The resulting object, an enhancement of $X$, is called a rough path (resp. model in the context of regularity structures).

There are situations - notably the KPZ equation - when the Stratonovich solution diverges, whereas the correct and desired object is given by the Itô solution. From
a rough paths perspective, this amounts to adjusting ("renormalize") the higher levels of the aforementioned enhancement. Doing so in an algebraically consistent way is a highly non-trivial task, and was achieved in a rough path resp. regularity structure set in the afore-mentioned works. In particular, [BHZ19] develops the algebraic theory of renormalization exclusively for smooth models, to which our study of smooth rough paths is aligned.

In this paper, we look at the indefinite signature path $t \mapsto \mathbf{X}_{t}:=\operatorname{Sig}\left(\left.X\right|_{[0, t]}\right)$, also known as canonical rough path lift, of a $\mathbb{R}^{d}$-valued smooth path $X$, as the solution of the linear differential equation

$$
\begin{equation*}
\dot{\mathbf{X}}_{t}=\mathbf{X}_{t} \otimes \dot{X}_{t}, \quad \mathbf{X}_{0}=\mathbf{1} \in \mathcal{G}, \tag{1}
\end{equation*}
$$

with, as is well-known (see e.g. Lyo98, Sec. 2.1.1] or [FV10, Ch. 7]) $\mathcal{G}:=G\left(\mathbb{R}^{d}\right)=$ $\exp _{\otimes}(\mathfrak{g}) \subset T\left(\left(\mathbb{R}^{d}\right)\right)$, where $\mathfrak{g}:=\mathcal{L}\left(\left(\mathbb{R}^{d}\right)\right)=\mathbb{R}^{d} \oplus\left[\mathbb{R}^{d}, \mathbb{R}^{d}\right] \oplus \ldots$ denotes the space of Lie series. The same construction in the quotient (or level $-N$ truncated) algebra

$$
\left(T^{N}\left(\mathbb{R}^{d}\right), \otimes_{N}, \mathbf{1}\right)
$$

yields a finite-dimensional Lie group $\mathcal{G}^{N}:=G^{N}\left(\mathbb{R}^{d}\right)=\exp _{\otimes_{N}}\left(\mathfrak{g}^{N}\right) \subset T^{N}\left(\mathbb{R}^{d}\right)$ with Lie algebra $\mathfrak{g}^{N}:=\mathcal{L}^{N}\left(\mathbb{R}^{d}\right)$, the Lie polynomials of degree less equal $N$. This yields an increasing family of Lie algebras (resp. groups) with inclusion map $i$ (resp. $j$ ).


Cartan's classical development of a smooth $\mathfrak{g}^{N}$-valued path $\mathfrak{y}$ is precisely given by solving the differential equation

$$
\dot{\mathbf{X}}_{t}=\mathbf{X}_{t} \otimes_{N} \mathfrak{y}(t), \quad \mathbf{X}_{0}=\mathbf{1} \in \mathcal{G}^{N}
$$

Basic results on linear differential equations in finite dimensions guarantee a unique and global solution. This gives a well-defined "projective" way to solve for $\mathbf{X}_{t}=\mathbf{X}_{t} \otimes$ $\mathfrak{y}(t), \mathbf{X}_{0}=1 \in \mathcal{G}$, for any $\mathfrak{g}$-valued path $\mathfrak{y}$. This is a precise generalization of (1) and leads us to the class of smooth rough path\&1. In infinite-dimensional Lie group theory, solvability of such equations has led to the notion of regular Lie group [Mil84; although this is of no concern to us and we refer to [BDS16, BDS18] for the subtleties of infinite groups like $\mathcal{G}$.

Conversely, every smooth $\mathcal{G}^{N}$-valued path $\mathbf{X}_{t}$ is the Cartan development of

$$
\mathfrak{y}(t):=\left\langle\omega_{\mathbf{X}_{t}}, \dot{\mathbf{X}}_{t}\right\rangle=\mathbf{X}_{t}^{-1} \otimes \dot{\mathbf{X}}_{t}=\left.\partial_{h}\right|_{h=0} \mathbf{X}_{t, t+h}=: \dot{\mathbf{X}}_{t, t} \in \mathfrak{g}^{N}
$$

[^0]Here $\omega_{\mathbf{x}}$ is the $\mathfrak{g}^{N}$-valued (left invariant) Maurer-Cartan form, given at $\mathbf{x} \in G^{N}\left(\mathbb{R}^{d}\right)$, by

$$
\begin{equation*}
\omega_{\mathbf{x}}:=\mathbf{x}^{-1} \otimes d \mathbf{x} \tag{2}
\end{equation*}
$$

which can be viewed as left logarithmic derivative of the identity map of $G^{N}\left(\mathbb{R}^{d}\right)$. The reason we encounter the left invariant Maurer-Cartan form, rather than its right invariant counterpart $\left((d \mathbf{x}) \otimes \mathbf{x}^{-1}\right)$ can be traced back to the order of interated integrals in the definition of the signatures, i.e. $u_{1} \leq \cdots \leq u_{n}$ rather than $u_{n} \leq \cdots \leq u_{1}$. We recall also that the Maurer-Cartan form has appeared in previous works of rough paths on manifolds CDL15, as well as signature based shape analysis [CLT19. We shall see in $\$ 2.4$ and $\$ 3.3$, that renormalization of rough differential equations, in the spirit of [BCFP19], can much benefit from this geometric view point. (For earlier use in the context of renormalization see also (CQRV02]) In the context of renormalization of rough differential equations however, its use appears to be new.

The geometry of $\mathcal{G}$ encodes validity of a chain rule, equivalently expressed in terms of shuffle identities, that in turn exhibits $\mathcal{G}$ as a character group of the shuffle Hopf algebra. This suggests correctly that the Maurer-Cartan perspective is not restrictive to $G\left(\mathbb{R}^{d}\right)$-valued ("geometric") rough paths, but valid for "general" rough paths, in the sense of [TZ20], with values in the character groups of a general graded Hopf algebra. However, too much generality does not allow for some of the concrete applications we have in mind, notably an understanding of differential equations driven by rough paths and their renormalization theory. We thus commence in Chapter 2 with smooth instances of geometric rough paths (in short: grp), which can be thought of as a multidimensional path enhanced with iterated integrals with classical integration by parts (shuffle) relations, together with suitable analytic conditions. A typical (non-smooth) example is given by Brownian motion with iterated integrals in the Stratonovich sense, see e.g.[FH20, Sec. 2.2] for precise definitions, see also Lyo98, LCL07, FV10, HK15, FH20. (We will review what we need in the main text below, cf. Definitions 2.1 and 2.3.) Another aspect concerns the sub-Riemannian structure of $G^{N}\left(\mathbb{R}^{d}\right)$, the state space of (level- $N$ ) geometric rough paths, see e.g. in [FV10, Remark 7.43] or FG16].

The indefinite signature of a $\mathbb{R}^{d}$-valued path always stays tangent to the left-invariant vector fields generated by the $d$ coordinate vector fields. In sub-Riemannian geometry, such paths are called horizontal. The study of smooth geometric rough paths is effectively the study of (possibly non-horizontal) smooth paths on $\mathcal{G}$. It is classical in geometric rough path theory to equip this space with the Carnot-Caratheodory metric. A generic smooth geometric rough path then has infinite length and are thus a genuine and interesting example of rough paths. We then continue to extend smooth rough paths in a more general setting:

Quasi-geometric theory: A quasi-geometric rough path should then be thought of as a multidimensional path enhanced with iterated integrals with classical integration by parts (shuffle) relation replaced by a generalized integration by parts rule known as quasishuffle. A typical example is given by Brownian motion with iterated integrals in the Itô sense. This structure also arises naturally when dealing with discrete sums (cf. [DET20] for signature sums) or piecewise constant paths, as well as Lévy processes [CEMW14, general semimartingales and finally rough path analysis [BCE20, Bel20. We also note unpublished presentations on quasi-rough paths by D. Kelly, who first introduced the concept, following his work [Kel12, HK15]. The very influential article [HK15] focused
on the interplay between geometric and branched rough paths, introduced in [Gub10], not central to this work. We note that quasi-shuffle structures have emerged independently in renormalization theory, see e.g. [Kre00, MP18, MP10, CGPZ20] and the references therein.

Hopf algebra constructions: We finally revisit the previous constructions from a general Hopf algebra construction. In view of [BCFP19] we do not single out the case of branched rough paths. Our graded approach to treat first geometric (Chapter 2), then quasi-geometric (Chapter 3) and finally the Hopf algebra case (Chapter 4) is a choice we made for essentially two reasons: (i) The material of Chapter 2 remains accessible to readers with a minimum on prerequisites and is also the setting that is most used in the context of signatures, including the recent developments in algebraic-geometric. (ii) Not every results obtained in the (quasi)geometric setting has a precise counter-part in the Hopf algebraic generality. For instance, as already observed in [BCFP19] in the context of branched rough paths, one loses certain uniqueness properties of renormalization operators when passing to more general structures.

Let us list the main contributions of this work with some detailed pointers to the main text.

- With Definitions 2.1, also 3.8, 4.1 we introduce the class of smooth rough paths in their respective setting and show in Theorems 2.8, also 3.10, 4.2 that level $-N$ smooth rough paths have a lift whose uniqueness hinges on some algebraic/geometric minimality. (This is in contrast to the classical Lyons lift of level $-N$ rough paths, where uniqueness depends on analytical conditions, see e.g. [FV10, Ch. 9].) We note that a related minimality condition appeared in the context of rough paths with jumps [FS17.
- An interesting insight is then that smooth rough paths, the resulting space of which is by nature non-linear, can be given a canonical linear structure. (This should be contrasted with adhoc linearizations based on Lyons-Victoir extension, see e.g. [FH20, Ex. 2.4] and especially [TZ20].)
- We finally revisit differential equations driven by our classes of rough paths, followed by their renormalization as initiated in [BCFP19]. Specifically, in Theorem 2.26 we highlight the role of smooth rough paths in the argument. Our subsequent extensions in Section 3 and 4 complements (and differ from) existing results, notably BCFP19, with a sole focus on geometric and branched structures, and related works [BCE20, Bru20] that involve/pass through branched constructions.

Acknowledgment. CB, PKF and SP were supported in part by DFG Research Unit FOR2402. PKF and RP were supported in part by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 683164). RP would like to thank Terry Lyons for a discussion that led to fundamental ideas for Section 2.2, Corollaries 2.29, 3.14, 3.25 and Section 4.3,

## 2 Smooth geometric rough paths

### 2.1 Definitions and fundamental properties

Let $\left(T\left(\left(\mathbb{R}^{d}\right)\right),+, \otimes\right)$ denote the algebra of tensor series over $\mathbb{R}^{d}$, equipped with the standard basis $e_{1}, . ., e_{d}$. Elements of $T\left(\left(\mathbb{R}^{d}\right)\right)$ are of the form

$$
\mathbf{x}=\sum \mathbf{x}^{w} e_{w}
$$

with summation over all words $w=\ell_{1} \cdots \ell_{n}$ with letters $\ell_{j} \in\{1, \ldots, d\}$, scalars $\mathbf{x}^{w}$ and $e_{w}:=e_{l_{1}} \otimes \cdots \otimes e_{l_{n}}$. The summation includes also 1 , the empty word. There is a natural pairing of $T\left(\left(\mathbb{R}^{d}\right)\right)$ with $T\left(\mathbb{R}^{d}\right)$, the space of tensor polynomial linearly spanned by the $e_{w}$, so that

$$
\begin{equation*}
\langle\mathbf{x}, w\rangle:=\left\langle\mathbf{x}, e_{w}\right\rangle=\mathbf{x}^{w}, \tag{3}
\end{equation*}
$$

and the same pairing applies to the truncated spaces $T^{N}\left(\mathbb{R}^{d}\right)$, consisting of tensor polynomials of degree at most $N \in \mathbb{N}$, spanned by pure tensors $e_{w}$ whose word $w$ has length $|w| \leq$ $N$. We denote the the canonical projection onto $T^{N}\left(\mathbb{R}^{d}\right)$ as $\operatorname{proj}_{N}: T\left(\left(\mathbb{R}^{d}\right)\right) \rightarrow T^{N}\left(\mathbb{R}^{d}\right)$. Equivalently, we can introduce $T^{N}\left(\mathbb{R}^{d}\right)$ as a quotient algebra. Indeed, introducing the ideal $T^{>N}\left(\left(\mathbb{R}^{d}\right)\right):=\bigoplus_{n>N}^{\infty}\left(\mathbb{R}^{d}\right)^{\otimes n}$ one immediately sees

$$
T^{N}\left(\mathbb{R}^{d}\right) \cong T\left(\left(\mathbb{R}^{d}\right)\right) / T^{>N}\left(\left(\mathbb{R}^{d}\right)\right)
$$

We write $\otimes_{N}$ for the induced "truncated tensor product".
Using the identification between words and tensors, we denote by m the shuffle product on words $\amalg: T\left(\mathbb{R}^{d}\right) \times T\left(\mathbb{R}^{d}\right) \rightarrow T\left(\mathbb{R}^{d}\right)$, defined by $w 山 \mathbf{1}=\mathbf{1} ш w=w$ for any word $w$ and the recursive definition

$$
\begin{equation*}
w i \amalg v j=(w \amalg v j) i+(v i \amalg w) j, \tag{4}
\end{equation*}
$$

for any couple of words $w, v$ and letters $i, j \in\{1, \ldots, d\}$. The product $\amalg$ induces a commutative algebra ${ }^{2}$ on $T\left(\mathbb{R}^{d}\right)$.

There are several Lie algebras we want to look at which stem from the algebra $\left(T\left(\left(\mathbb{R}^{d}\right)\right), \otimes\right)$; we use the notation $\mathcal{L}\left(\mathbb{R}^{d}\right)$ for the Lie algebra generated by the letters $\mathbb{R}^{d}$ (the space of Lie polynomials), $\mathcal{L}^{N}\left(\mathbb{R}^{d}\right)$ for the projection of $\mathcal{L}\left(\mathbb{R}^{d}\right)$ into $T^{N}\left(\mathbb{R}^{d}\right)$, and $\mathcal{L}\left(\left(\mathbb{R}^{d}\right)\right) \subset T\left(\left(\mathbb{R}^{d}\right)\right)$ the Lie series. We define the tensor and truncated tensor exponential $\exp _{\otimes}: T^{>0}\left(\left(\mathbb{R}^{d}\right)\right) \rightarrow T\left(\left(\mathbb{R}^{d}\right)\right), \exp _{\otimes_{N}}: T^{N}\left(\mathbb{R}^{d}\right) \cap T^{>0}\left(\left(\mathbb{R}^{d}\right)\right) \rightarrow T^{N}\left(\mathbb{R}^{d}\right)$ given respectively by

$$
\begin{equation*}
\exp _{\otimes} \mathbf{x}=\sum_{n \geq 0} \frac{\mathbf{x}^{\otimes n}}{n!}, \quad \exp _{\otimes_{N}} \mathbf{x}=\sum_{n=0}^{N} \frac{\mathbf{x}^{\otimes_{N} n}}{n!} \tag{5}
\end{equation*}
$$

Then it is well known ( $\mathbb{R e u 9 3}$, LCL07] $)$ that $G\left(\mathbb{R}^{d}\right)=\exp _{\otimes} \mathcal{L}\left(\left(\mathbb{R}^{d}\right)\right)$ is a group with operation $\otimes$ which satisfies

$$
\begin{equation*}
G\left(\mathbb{R}^{d}\right)=\left\{\mathbf{x} \in T\left(\left(\mathbb{R}^{d}\right)\right):\langle\mathbf{x}, \mathbf{1}\rangle=1,\langle\mathbf{x}, v ш w\rangle=\langle\mathbf{x}, v\rangle\langle\mathbf{x}, w\rangle \text { for all words } w, v\right\} \tag{6}
\end{equation*}
$$

[^1]Similar results hold for $G^{N}\left(\mathbb{R}^{d}\right)$, now in terms of words with joint length $\leq N$ and as the image of $\mathcal{L}^{N}\left(\mathbb{R}^{d}\right)$ under $\exp _{\otimes_{N}}$ respectively. We note that $\left(G^{N}\left(\mathbb{R}^{d}\right), \otimes_{N}\right)$ is a bona fide finite-dimensional Lie group with Lie algebra $\mathcal{L}^{N}\left(\mathbb{R}^{d}\right)$.

The following definitions are standard (as e.g. found in HK15), but with analytic Hölder / variation type conditions replaced by a smoothness assumptions.

Definition 2.1. We call level- $N$ smooth geometric rough path (in short: $N$-sgrp) over $\mathbb{R}^{d}$ any non-zero path $\mathbf{X}:[0, T] \rightarrow T^{N}\left(\mathbb{R}^{d}\right)$ such that
(a.i) The shuffle relation holds for all times $t \in[0, T]$

$$
\begin{equation*}
\left\langle\mathbf{X}_{t}, v ш w\right\rangle=\left\langle\mathbf{X}_{t}, v\right\rangle\left\langle\mathbf{X}_{t}, w\right\rangle \tag{7}
\end{equation*}
$$

for all words with joint length $|v|+|w| \leq N$.
(a.ii) For every word of length $|w| \leq N$, the map $t \mapsto\left\langle\mathbf{X}_{t}, w\right\rangle$ is smooth. We write

$$
\dot{\mathbf{X}}_{t}=\sum_{|w| \leq N}\left\langle\dot{\mathbf{X}}_{t}, w\right\rangle e_{w}
$$

for the derivative of $\mathbf{X}$.
By smooth geometric rough path (in short: sgrp) we mean a path with values in $T\left(\left(\mathbb{R}^{d}\right)\right)$ with all defining " $\leq N$ " restrictions on the word's length omitted.

Remark 2.2. By Proposition 2.14 below $N$-sgrp are smooth $G^{N}\left(\mathbb{R}^{d}\right)$-valued paths, thus (with respect to the appropriate Carnot-Caratheodory metric [FV10]) genuine $1 / \mathrm{N}$ Hölder regular rough paths, which justifies our terminology. Similarly, sgrp's are nothing but smooth $G\left(\mathbb{R}^{d}\right)$-valued paths, provided $G\left(\mathbb{R}^{d}\right)$ is equipped with a suitable "weak" differential structure to make it a topological Lie group, see BDS18.

The shuffle relation applied to empty words and the demand that $\mathbf{X}$ is non-zero imply together with continuity in $t$ that $\left\langle\mathbf{X}_{t}, \mathbf{1}\right\rangle=1$ for all $t \in[0, T]$, hence $\mathbf{X}$ is (similar to formal power series) invertible with respect to $\otimes_{N}$. Every $N$-sgrp gives then rise to increments, $(s, t) \mapsto \mathbf{X}_{s}^{-1} \otimes_{N} \mathbf{X}_{t}$. This motivates the following definition.

Definition 2.3. We call level- $N$ smooth geometric rough model (in short: $N$-sgrm) over $\mathbb{R}^{d}$ any non-zero map $\mathbf{X}:[0, T]^{2} \rightarrow T^{N}\left(\mathbb{R}^{d}\right)$ such that
(b.i) the shuffle relation (7) holds with $\mathbf{X}_{t}$ replaced by $\mathbf{X}_{s, t}$, any $s, t$.
(b.ii) Chen's relation holds, by which we mean

$$
\begin{equation*}
\mathbf{X}_{s u} \otimes_{N} \mathbf{X}_{u t}=\mathbf{X}_{s, t} \tag{8}
\end{equation*}
$$

for any $s, u, t \in[0, T]$.
(b.iii) For every word of length $|w| \leq N$, the map $\mapsto\left\langle\mathbf{X}_{s, t}, w\right\rangle$ is smooth, for one (equivalently: all) base point(s) $s \in[0, T]$.

By smooth geometric rough model (in short sgrm) we mean a map with values in $T\left(\left(\mathbb{R}^{d}\right)\right)$, with all " $\leq N$ " quantifiers omitted and equation (8) with $\otimes_{N}$ replaced by $\otimes$.

Remark 2.4. The terminology "model" in is consistent with Hairer's regularity structures. More specifically, given a $N$-sgrm $\mathbf{X}=\mathbf{X}_{s, t}$ we have the map

$$
s \mapsto\left\{T^{N}\left(\mathbb{R}^{d}\right) \ni \mathbf{u} \mapsto\left\langle\mathbf{X}_{s,}, \mathbf{u}\right\rangle\right\}
$$

which together with Chen's relation yields precisely a model in the sense of regularity structures. See e.g. [FH20, Sec. 13.2.2], Pre16, Thm. 5.15] and [BCFP19, Proposition 48]. Our use of the adjective "smooth" is also consistent with the notion of smooth model, used by Hairer and coworkers and central to the algebraic renormalization theory of (BHZ19].

It follows from (7) resp. (b.i) and the non-zero demand that $N$-smooth geometric rough paths and models really take values in $G^{N}\left(\mathbb{R}^{d}\right)$; similarly for (tensor series) smooth rough paths and models and $G\left(\mathbb{R}^{d}\right)$. Clearly, every level- $N$ smooth geometric rough path $\mathbf{X}:[0, T] \rightarrow T^{N}\left(\mathbb{R}^{d}\right)$ induces a level- $N$ smooth geometric rough model $\mathbf{X}:[0, T]^{2} \rightarrow$ $T^{N}\left(\mathbb{R}^{d}\right)$ in the above sense, by considering the increments

$$
\begin{equation*}
\mathbf{X}_{s, t}:=\mathbf{X}_{s}^{-1} \otimes_{N} \mathbf{X}_{t} . \tag{9}
\end{equation*}
$$

Conversely, every level- $N$ smooth geometric rough model defines a level $N$ smooth geometric rough path $\mathbf{X}_{t}:=\mathbf{X}_{0, t}$ so that $N$-sgrp and $N$-sgrm are equivalent modulo a starting point in $G^{N}\left(\mathbb{R}^{d}\right)$.

Definition 2.5. A $\operatorname{sgrp} \mathbf{X}$ is called extension of some $N$-sgrp $\mathbf{Y}$ if

$$
\left\langle\mathbf{X}_{t}, w\right\rangle=\left\langle\mathbf{Y}_{t}, w\right\rangle \quad \text { for all } t \text { and whenever }|w| \leq N
$$

if this holds for a $N^{\prime}$-sgrp $\mathbf{X}$, with $N<N^{\prime}<\infty$, we call it $N^{\prime}$-extension of $\mathbf{Y}$. We adopt also the same denomination if $\mathbf{Y}$ is a $N$-sgrm and $\mathbf{X}$ is a $\operatorname{sgrm}\left(N^{\prime}\right.$-sgrm) and one has the same relation above for any $s, t \in[0, T]$.

When $d=1$, the situation is trivial, and sgrp's are in one-to-one correspondence with smooth scalar paths: An arbitrary scalar path $Y$ with initial point $Y_{0}=0$ has a (unique) extension given by

$$
\mathbf{X}_{t}=\mathbf{1}+Y_{t} e_{1}+\frac{\left(Y_{t}\right)^{2}}{2!} e_{11}+\ldots=\exp _{\otimes} Y_{t} \in T((\mathbb{R}))
$$

Conversely, every sgrp with $\mathbf{X}_{0}=\mathbf{1}$ must be of this form, as a consequence of the shuffle relations. When $d>1$ such extensions are never unique. For instance, given the two basis vectors $e_{1}, e_{2} \in \mathbb{R}^{2}$ and denoting by $\mathbf{0}$ the common zero of $\left(\mathbb{R}^{d}\right)^{\otimes n}, n \geq 1$ we see that $t \mapsto(1, \mathbf{0}, \mathbf{0}) \in T^{2}\left(\mathbb{R}^{2}\right)$ and $t \mapsto\left(1, \mathbf{0}, t\left[e_{1}, e_{2}\right]\right) \in T^{2}\left(\mathbb{R}^{2}\right)$ are both 2 -sgrps over $\mathbb{R}$, and hence level-2 extensions of the trivial 1 -sgrp $t \mapsto(1, \mathbf{0})$.

In classical rough path analysis Lyo98, FV10, a graded $p$-variation (or Hölder) condition is enforced, which guarantees a unique extension. We give here a novel algebraic condition that enforces uniqueness, somewhat similar in spirit to the minimal jump extension of cadlag rough paths in [FS17. This condition is motivated by the following result.

Proposition 2.6. Given a sgrm $\mathbf{X}$ over $\mathbb{R}^{d}$, for all times $s$ one has,

$$
\begin{equation*}
\dot{\mathbf{X}}_{s, s}:=\left.\partial_{t}\right|_{t=s} \mathbf{X}_{s, t} \in \mathcal{L}\left(\left(\mathbb{R}^{d}\right)\right) \tag{10}
\end{equation*}
$$

The analogue statement holds for $N$-sgrm's, with $\mathcal{L}\left(\left(\mathbb{R}^{d}\right)\right)$ replaced by its truncation $\mathcal{L}^{N}\left(\mathbb{R}^{d}\right)$. We call the path in (10) the diagonal derivative of $\mathbf{X}$.

Proof. A geometric proof is not difficult. We define the $G^{N}\left(\mathbb{R}^{d}\right)$-valued path $\mathbf{X}_{t}:=\mathbf{X}_{0, t}$ so that $\dot{\mathbf{X}}_{s, s}:=\mathbf{X}_{s}^{-1} \otimes \dot{\mathbf{X}}_{s}$ which is exactly the Maurer-Cartan form at $\mathbf{X}_{s}$, evaluated at the tangent vector $\dot{\mathbf{X}}_{s}$. This results in an element in the Lie algebra, which here is identified with $\mathcal{L}^{N}\left(\mathbb{R}^{d}\right)$. Since $N$ is abitrary, this also implies the first claim.

Remark 2.7. Let us illustrate the previous Proposition in case $N=2$. In view of the defining shuffle relation of sgrm's, applied to single letter words $i$ and $j$,

$$
\left\langle\mathbf{X}_{s, t}, i\right\rangle\left\langle\mathbf{X}_{s, t}, j\right\rangle=\left\langle\mathbf{X}_{s, t}, i j\right\rangle+\left\langle\mathbf{X}_{s, t}, j i\right\rangle
$$

Dividing this identity by $t-s>0$, followed by sending $t \downarrow s$, one has immediately

$$
\begin{equation*}
\left\langle\dot{\mathbf{X}}_{s, s}, i j\right\rangle+\left\langle\dot{\mathbf{X}}_{s, s}, j i\right\rangle=0 \tag{11}
\end{equation*}
$$

so that the second tensor level of $\dot{\mathbf{X}}_{s, s}$ is anti-symmetric, hence $\dot{\mathbf{X}}_{s, s} \in \mathbb{R}^{d} \oplus\left[\mathbb{R}^{d}, \mathbb{R}^{d}\right]$.
Theorem 2.8 (Fundamental Theorem of sgrm). Given an $N$-sgrm $\mathbf{Y}$ for some $N \in \mathbb{N}$, there exists exactly one sgrm extension $\mathbf{X}$ of $\mathbf{Y}$ which is minimal in the sense that for all $s \in[0, T]$ one has

$$
\dot{\mathbf{X}}_{s, s} \in \mathcal{L}^{N}\left(\mathbb{R}^{d}\right) \subset \mathcal{L}\left(\left(\mathbb{R}^{d}\right)\right)
$$

This unique choice then in fact satisfies $\dot{\mathbf{X}}_{s, s}=\dot{\mathbf{Y}}_{s, s}$. We call $\operatorname{MinExt}(\mathbf{Y}):=\mathbf{X}$ the minimal extension of $\mathbf{Y}$ and also $\operatorname{MinExt}^{\mathrm{N}^{\prime}}(\mathrm{Y}):=\operatorname{proj}_{N^{\prime}} \mathbf{X}$, for $N^{\prime}>N$, the $N^{\prime}$ minimal extension of $\mathbf{Y}$. For a fixed interval $[s, t] \subset[0, T]$ it holds that $\mathbf{X}_{s, t}$ only depends on $\left\{\left.\mathbf{Y}\right|_{[u, v]}: s \leq u \leq v \leq t\right\}$ and we introduce the signature of $\mathbf{Y}$ on $[s, t]$ by

$$
\operatorname{Sig}\left(\left.\mathbf{Y}\right|_{[s, t]}\right):=\mathbf{X}_{s, t} \in G\left(\mathbb{R}^{d}\right)
$$

Proof. (Existence) Thanks to the previous proposition

$$
\mathfrak{y}(t):=\dot{\mathbf{Y}}_{t, t} \in \mathcal{L}^{N}\left(\mathbb{R}^{d}\right)
$$

defines a smooth path with values in the Lie algebra $\mathcal{L}^{N}\left(\mathbb{R}^{d}\right)$. Its Cartan development into $G\left(\mathbb{R}^{d}\right)$ amounts to solve for $\dot{\mathbf{X}}_{t}=\mathbf{X}_{t} \otimes \mathfrak{y}(t), \mathbf{X}_{0}=\mathbf{1} \in G\left(\mathbb{R}^{d}\right)$. It is enough to do this in finite dimensions, say in $G^{N^{\prime}}\left(\mathbb{R}^{d}\right)$ for arbitrary $N^{\prime}>N$, in which case such differential equations have a unique and global solution. Indeed, existence of a unique local solution is clear from ODE theory, whereas non-explosion is a consequence of linearity of this equation. See also a much more general reference [IN99]. Since the natural projections from $G^{N^{\prime}+1}\left(\mathbb{R}^{d}\right) \rightarrow G^{N^{\prime}}\left(\mathbb{R}^{d}\right)$ are Lie group morphisms, we obtain a consistent family of $N^{\prime}$-extensions which defines $t \mapsto \mathbf{X}_{t}$ in the projective limit. The minimal extension is then given by $\mathbf{X}_{s, t}=\mathbf{X}_{s}^{-1} \otimes \mathbf{X}_{t}$, where we note that

$$
\dot{\mathbf{X}}_{t, t}=\mathbf{X}_{t}^{-1} \otimes \dot{\mathbf{X}}_{t}=\mathfrak{y}(t) \in \mathcal{L}^{N}\left(\mathbb{R}^{d}\right)
$$

(Uniqueness) It is sufficient to consider $N^{\prime}=N+1$. In this case two level- $N^{\prime}$ extensions $\mathbf{X}, \overline{\mathbf{X}}$, differ by an element in the center of $G^{N^{\prime}}\left(\mathbb{R}^{d}\right)$, so that $\Psi_{s, t}:=\left\langle\mathbf{X}_{s, t}-\overline{\mathbf{X}}_{s, t}, w\right\rangle$ is additive for every word $|w| \leq N+1$. We need to show that $\Psi \equiv 0$. By assumption, $\Psi$ vanishes on words of length $|w| \leq N$, so we can assume $|w|=N^{\prime}=N+1$. With $\Psi_{t}:=\Psi_{0, t}$ note that $\Psi_{s, t}=\Psi_{t}-\Psi_{s}$. Write $\mathbf{X}_{t}=\mathbf{X}_{0, t}$, and similarly for $\overline{\mathbf{X}}$, and also $\sim$ for equality in the limit $t \downarrow s$. Then

$$
\dot{\Psi}_{s} \sim \frac{\Psi_{s, t}}{(t-s)}=\frac{\left\langle\mathbf{X}_{s}^{-1} \otimes\left(\mathbf{X}_{t}-\mathbf{X}_{s}\right)-\overline{\mathbf{X}}_{s}^{-1} \otimes\left(\overline{\mathbf{X}}_{t}-\overline{\mathbf{X}}_{s}\right), w\right\rangle}{(t-s)} \sim\left\langle\mathbf{X}_{s}^{-1} \otimes \dot{\mathbf{X}}_{s}-\overline{\mathbf{X}}_{s}^{-1} \otimes \dot{\mathbf{X}}_{s}, w\right\rangle
$$

and in view of the minimality assumption of order $N$ of both $\mathbf{X}, \overline{\mathbf{X}}$, with the condition $|w|=N+1$, we see that $\dot{\Psi}$ is zero, hence $\Psi \equiv \Psi_{0}=0$ which concludes the argument for uniqueness.

Remark 2.9 (Computing the minimal extension). Note that the existence part of this proof is constructive and gives the minimal extension via solving a linear differential equation. To make this explicit in case of a $N$-sgrp $Y$, it suffices to solve

$$
\dot{\mathbf{X}}=\mathbf{X} \otimes\left(\mathbf{Y}^{-1} \otimes_{N} \dot{\mathbf{Y}}\right), \quad \mathbf{X}_{0}=\mathbf{1} \in T\left(\left(\mathbb{R}^{d}\right)\right)
$$

To compute the signature on $[s, t]$, it suffices to start at the initial condition $\mathbf{X}_{s}=\mathbf{1}$.
It follows automatically from the proof of Theorem 2.8 that, defining $\operatorname{MinExt}(N)$ as the class of sgrms which arise as minimal extension of some $N$-sgrms over $\mathbb{R}^{d}$, we have the inclusions

$$
\begin{gathered}
\operatorname{MinExt}(1) \subset \operatorname{MinExt}(2) \subset \cdots \subset \operatorname{MinExt}(N) \subset \cdots \subset\left\{\operatorname{sgrm} \text { over } \mathbb{R}^{d}\right\}, \\
\bigcup_{N \geq 1} \operatorname{MinExt}(N) \subset\left\{\operatorname{sgrm} \text { over } \mathbb{R}^{d}\right\}
\end{gathered}
$$

When $d>1$, all inclusions above are strict. For instance, take a non-finite Lie series, i.e. $\mathfrak{v} \in \mathcal{L}\left(\left(\mathbb{R}^{d}\right)\right) \backslash \mathcal{L}\left(\mathbb{R}^{d}\right)$. Then $\mathbf{X}_{s, t}=\exp _{\otimes}(\mathfrak{v}(t-s))$ defines a sgrm, which is not a minimal extension of any $N$-sgrm. These strict inclusions motivate the following finer subset of sgrms.

Definition 2.10. A sgrm $\mathbf{X}$ over $\mathbb{R}^{d}$ is called a good $\operatorname{sgrm}$ if $\mathbf{X}=\operatorname{MinExt}(\mathbf{Y})$ for some $N$-sgrm $\mathbf{Y}, N \in \mathbb{N}$.

Recalling that a minimal extension has the same diagonal derivative as the underlying $N$-sgrm, one has the immediate characterisation of good sgrms.

Lemma 2.11. $A \operatorname{sgrm} \mathbf{X}$ is good if and only if for all times $s \in[0, T]$ one has

$$
\begin{equation*}
\dot{\mathbf{X}}_{s, s} \in \mathcal{L}\left(\mathbb{R}^{d}\right) . \tag{12}
\end{equation*}
$$

We link the notion of minimal extension with previous constructions in the literature.
Example 2.12. Every smooth path $Y:[0, T] \rightarrow \mathbb{R}^{d}$ can be regarded (somewhat trivially) as 1 -sgrp $\mathbf{Y}=(1, Y)$. The solution to

$$
\dot{\mathbf{X}}=\mathbf{X} \otimes\left(\mathbf{Y}^{-1} \otimes_{1} \dot{\mathbf{Y}}\right)=\mathbf{X} \otimes \dot{Y}, \quad \mathbf{X}_{0}=\mathbf{1} \in T\left(\left(\mathbb{R}^{d}\right)\right)
$$

is then precisely given by the stack of iterated integrals

$$
\mathbf{X}_{t}=\operatorname{Sig}\left(\left.Y\right|_{[0, t]}\right)=\left(1, \int d Y, \ldots, \int(d Y)^{\otimes n}, \ldots\right) \in T\left(\left(\mathbb{R}^{d}\right)\right)
$$

with $n$-fold integration over the $n$-simplex over $[0, t]$, i.e. $0 \leq u_{1} \leq \cdots \leq u_{n} \leq t$. The signature of $\mathbf{Y}$ on $[s, t]$ coincides then with the usual definition of signature of $Y$, see e.g. [LCL07], modulo a choice of unit initial data at time $s$, which entails integration over simplices over $[s, t]$.

Example 2.13. Fix a step-2 Lie element over $\mathbb{R}^{d}$,

$$
\mathfrak{v}=(\mathfrak{a}, \mathfrak{b}) \in \mathbb{R}^{d} \oplus\left[\mathbb{R}^{d}, \mathbb{R}^{d}\right] \equiv \mathcal{L}^{2}\left(\mathbb{R}^{d}\right)
$$

where $\left[\mathbb{R}^{d}, \mathbb{R}^{d}\right]$ stands for the set of anti-symmetric 2-tensors over $\mathbb{R}^{d}$. Consider the example of a 2 -sgrm given by

$$
\mathbf{Y}_{s, t}=\left(1, \mathfrak{a}(t-s), \mathfrak{b}(t-s)+\frac{1}{2} \mathfrak{a}^{\otimes 2}(t-s)^{2}\right)=\exp _{\otimes_{2}}(\mathfrak{v}(t-s)) \in T^{2}\left(\mathbb{R}^{d}\right)
$$

One easily sees that $\dot{\mathbf{Y}}_{t, t}=\mathfrak{v}$ for all $t \in[0, T]$. Since $\mathfrak{v}$ is constant in time, the differential equation

$$
\dot{\mathbf{X}}=\mathbf{X} \otimes \mathfrak{v}, \quad \mathbf{X}_{s}=1 \in T\left(\left(\mathbb{R}^{d}\right)\right)
$$

has an explicit exponential solution at any time $t$ given by

$$
\begin{equation*}
\mathbf{X}_{s, t}=\exp _{\otimes}(\mathfrak{v}(t-s))=1+\mathfrak{v}(t-s)+\frac{1}{2} \mathfrak{v}^{\otimes 2}(t-s)^{2}+\cdots \in T\left(\left(\mathbb{R}^{d}\right)\right) \tag{13}
\end{equation*}
$$

This is precisely the signature of $\mathbf{Y}$ on $[s, t]$ and $\mathbf{X}$ is the minimal extension of $\mathbf{Y}$. Modulo a starting point in the group such a construction is well-known as a log-linear rough path, here the special case of second order logarithm. The case $\mathfrak{v}=(\mathfrak{a}, 0)$ is covered in Example 2.12 via the constant velocity path $\dot{Y} \equiv \mathfrak{a}$. By taking $\mathfrak{v}=(0, \mathfrak{b})$ the minimal extension coincides with a pure area rough path, see [Gal19] for an algebraic geometric perspective on these structures. The explicit exponential solution (13) is possible here thanks to the constant velocity $\mathfrak{v}$. In a general situation, with time-dependent $\mathfrak{v}=\mathfrak{v}(t)$ the solution has exponential form given by Magnus expansion and additional commutator terms will appear, see e.g. [IN99.

We finish this subsection by relating sgrp to (weakly) geometric rough paths in the sense of standard definitions as found e.g. in [FV10, HK15].

Proposition 2.14. One has the following properties:
(i) Every $N$-sgrp $\mathbf{X}$ is a weakly $1 / N$-Hölder weakly geometric rough path. Consequently, $\mathbf{X}$ is also a $\gamma$-Hölder geometric rough path, for any $1 / \gamma \in(N, N+1)$, and the set of $N$-sgrp is dense therein.
(ii) The minimal extension of some $N$-sgrp $\mathbf{X}$ to a coincides with the Lyons lift of $\mathbf{X}$, as constructed e.g. in [FV10, Ch. 9].

Proof. (i) We only need to discuss the analytic regularity, which is formulated for the increments $\mathbf{X}_{s, t}=\mathbf{X}^{-1} \otimes \mathbf{X}_{t}$, i.e. the associated rough model. Thanks to (b.iii), we have $\left|\left\langle\mathbf{X}_{s, t}, w\right\rangle\right| \lesssim|t-s| \lesssim|t-s|^{|w|(1 / N)}$ uniformly over $s, t \in[0, T]$, using that $|w| \leq N$. The consequence follows from well-known relations between geometric and weakly geometric rough paths [FV10, Ch. 9]. The final density statement is clear, since already the minimal level $-N$ extensions of smooth paths (a.k.a. canonical lifts) are dense in this level- $N$ rough paths space, cf. [FV10].
(ii) We remark that the minimal extension of $\mathbf{X}$ can be constructed as solving a linear differential equation driven by $\mathbf{X}$, in the sense of Definition 2.18 below, whereas the Lyons lift solves the analogous rough differential equations.

### 2.2 Canonical sum and minimal coupling of smooth geometric rough models

As was pointed out, the state space of a sgrp is $G\left(\mathbb{R}^{d}\right)$, a non-linear subset of $T\left(\left(\mathbb{R}^{d}\right)\right)$. In particular, the pointwise sum $\mathbf{X}_{t}+\mathbf{Y}_{t}$ of any given two sgrps $\mathbf{X}, \mathbf{Y}$ does not generally make sense as a sgrp. From a classical rough path perspective, the obstruction to making sense of the addition of rough path increments lies in its possible dependence on (a priori) missing mixed iterated integrals. For smooth rough paths, however, it turns out that there is a canonical way to add smooth geometric rough models and, more generally a scalar multiplication.

Definition 2.15. For any fixed sgrms $\mathbf{X}, \mathbf{Y}$ let $t \mapsto \mathbf{Z}_{t} \in T\left(\left(\mathbb{R}^{d}\right)\right)$ be the Cartan development of $\dot{\mathbf{X}}_{s, s}+\dot{\mathbf{Y}}_{s, s}$, i.e. the unique solution to

$$
\dot{\mathbf{Z}}_{t}=\mathbf{Z}_{t} \otimes\left(\dot{\mathbf{X}}_{t, t}+\dot{\mathbf{Y}}_{t, t}\right), \quad \mathbf{Z}_{0}=\mathbf{1}
$$

We then write $\mathbf{Z}:=\mathbf{X} \boxplus \mathbf{Y}$ for the associated sgrm and call it the canonical sum of $\mathbf{X}$ and $\mathbf{Y}$. For any $\lambda \in \mathbb{R}$ we define also the sgrm $\mathbf{Z}=\lambda \boxtimes \mathbf{X}$ via the Cartan development of $\lambda \dot{\mathbf{X}}_{s, s}$, we call it the canonical scalar multiplication.

We will see later that this addition of sgrm overlaps non-trivially with the renormalization of rough path. Operations $\square$ and $\boxplus$ equip the space of sgrps with a vector space structure and by Lemma 2.11 good sgrps form a linear subspace. For some general comments on these operations we refer to Remark 4.15 in the more general framework of roughs path associated to a Hopf algebra. We simply remark that in the smooth geometric setting the canonical sum coincides with a construction of Lyons for $p$-variation weakly geometric rough paths, see [Lyo98, Section 3.3.1 B].

Proposition 2.16. For any fixed couple of sgrms $\mathbf{X}, \mathbf{Y}$ a map $\mathbf{Z}:[0, T]^{2} \rightarrow G\left(\mathbb{R}^{d}\right)$ coincides with $\mathbf{X} \boxplus \mathbf{Y}$ if and only if $\mathbf{Z}$ satisfies $\mathbf{Z}_{s, t}=\mathbf{Z}_{s, u} \otimes \mathbf{Z}_{u, t}$ for $s, u, t \in[0, T]$ and one has for any $s \in[0, T]$

$$
\begin{equation*}
\mathbf{Z}_{s, t}=\mathbf{X}_{s, t} \otimes \mathbf{Y}_{s, t}+R_{s, t}, \tag{14}
\end{equation*}
$$

for some $R_{s, t} \in T\left(\left(\mathbb{R}^{d}\right)\right)$ such that for all $x \in T\left(\mathbb{R}^{d}\right)$ one has $\left\langle R_{s, t}, x\right\rangle=o(|t-s|)$ as $t \rightarrow s$. Moreover, we have the relations

$$
\begin{equation*}
\mathbf{X}_{s, t} \otimes \mathbf{Y}_{s, t}=\mathbf{Y}_{s, t} \otimes \mathbf{X}_{s, t}+r_{s, t}=\mathbf{X}_{s, t}+\mathbf{Y}_{s, t}-\mathbf{1}^{*}+r_{s, t}^{\prime} \tag{15}
\end{equation*}
$$

for some $r_{s, t}, r_{s, t}^{\prime} \in T\left(\left(\mathbb{R}^{d}\right)\right)$ such that for all $x \in T\left(\mathbb{R}^{d}\right)$ one has $\left\langle r_{s, t}, x\right\rangle,\left\langle r_{s, t}^{\prime}, x\right\rangle=$ $o(|t-s|)$ as $t \rightarrow s$.

Proof. See the more general Proposition 4.16.
Remark 2.17. Looking back at Lyo98, Section 3.3.1 B] and Proposition 2.14, we actually conclude that it is possible to sum a $N$-sgrm $\mathbf{X}$ to any general $\gamma$-Hölder weakly geometric rough path $\mathbf{W}$ for any $\gamma \in(0,1)$ in a canonical way, though the construction is not obtained via diagonal derivatives but via the demand $(\mathbf{X} \boxplus \mathbf{W})_{s, t}=\mathbf{X}_{s, t} \otimes \mathbf{W}_{s, t}+o(|t-s|)$ and a sewing lemma argument. This gives a strong motivation of looking at smooth rough paths as "universal perturbations" (accordingly to Lyo98, Section 3.3.1 B]) of $\gamma$-Hölder weakly geometric rough paths.

The canonical sum can now be used to define a minimal coupling in the situation where we have a finite number of smooth geometric rough models $\mathbf{X}^{i}:[0, T] \rightarrow G\left(\mathbb{R}^{d_{i}}\right)$. By putting $d=\sum_{i} d_{i}$ and fixing a canonical embedding $\mathbb{R}^{d_{i}} \subset \mathbb{R}^{d}$ we can uniquely construct an injective $\otimes$ homomorphisms $\iota_{i}: T\left(\left(\mathbb{R}^{d_{i}}\right)\right) \rightarrow T\left(\left(\mathbb{R}^{d}\right)\right)$ that extends the embedding and sends $G\left(\mathbb{R}^{d_{i}}\right)$ into $G\left(\mathbb{R}^{d}\right)$. Then we can consider the sgrms $\iota_{i} \mathbf{X}^{i}:[0, T] \rightarrow$ $G\left(\mathbb{R}^{d}\right)$ and we define the minimal coupling of the $\mathbf{X}^{i}$ as the canonical sum of the $\iota_{i} \mathbf{X}^{i}$

$$
\begin{equation*}
\left(\mathbf{X}^{1}, \ldots, \mathbf{X}^{m}\right)_{\min }:=\iota_{1} \mathbf{X}^{1} \boxplus \cdots \boxplus \iota_{m} \mathbf{X}^{m} \tag{16}
\end{equation*}
$$

The name minimal amounts to the fact that we are choosing a sgrm which involves the least possible information on mixed iterated integrals. This construction can also a partial generalization of Lyons-Young " $(p, q)$ " pairing discussed in [FV10].

### 2.3 Differential equations driven by SGRP

Let $\left(f_{1}, \ldots, f_{d}\right) \in\left(\operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)\right)^{d}$ be a collection of smooth vector fields, with bounded derivatives of all orders, so that all stated operations and differential equations below make sense. For the empty word 1 , set $f_{1}=i d$, the identity vector field. For a word $w=\ell_{1} \ldots \ell_{n}$ with $|w| \geq 1$ letters $\ell_{j} \in\{1, \ldots, d\}$, define the vector field

$$
\begin{equation*}
f_{w}:=f_{\ell_{1}} \triangleright\left(\ldots \triangleright\left(f_{\ell_{n-1}} \triangleright f_{\ell_{n}}\right) \ldots\right) \tag{17}
\end{equation*}
$$

where $\triangleright$ is the following operation of two vector fields: i.e. using Einstein summation over $i, j=1, \ldots, e$ we set

$$
\begin{equation*}
f \triangleright g:=f^{i}\left(\partial_{i} g^{j}\right) \partial_{j} \tag{18}
\end{equation*}
$$

where $f=\left(f^{i} \partial_{i}\right)$, and $g=\left(g^{j} \partial_{j}\right)$. Using tensor calculus notation, we can equivalently set $f \triangleright g=(\nabla g) f$ where $\nabla g$ is the Jacobian matrix of $g$. The resulting family of maps $\left\{y \mapsto f_{w}(y)\right\}_{w}$ is represented with a linear map $f: T\left(\mathbb{R}^{d}\right) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$ such that

$$
\begin{equation*}
T\left(\mathbb{R}^{d}\right) \in \mathbf{x} \mapsto f_{\mathbf{x}}(\cdot)=\langle f(\cdot), \mathbf{x}\rangle \tag{19}
\end{equation*}
$$

where the final pairing amounts to view $f$ as (formal) sum $\sum f_{w} e_{w}$, with summation over all words. Importantly, this map, restricted to Lie polynomials induces a Lie algebra morphism

$$
\mathcal{L}\left(\mathbb{R}^{d}\right) \ni \mathfrak{u} \mapsto f_{\mathfrak{u}} \in \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)
$$

which is uniquely determined by his values on the canonical basis of $\mathbb{R}^{d}$, since $\mathcal{L}\left(\mathbb{R}^{d}\right)$ is isomorphic to the free Lie algebra over $\mathbb{R}^{d}$. From an intrinsic geometric point of view, $f_{\mathfrak{u}}$ is a vector field over $\mathbb{R}^{e}$ interpreted as a manifold, with its true for $f_{\mathbf{x}}$ only to the extend that one implicitly uses that flat connection on $\mathbb{R}^{e}$.

In what follows, we equip $T\left(\mathbb{R}^{d}\right)$ with an inner product structure, by declaring orthonormal the basis vectors $e_{w} \in T\left(\mathbb{R}^{d}\right)$, induced by distinct words $w$. Viewing $T\left(\mathbb{R}^{d}\right)$ as subspace of $T\left(\left(\mathbb{R}^{d}\right)\right.$ ), this is consistent with the readily used pairing $\langle\mathbf{x}, \mathbf{z}\rangle$ with $\mathbf{x} \in T\left(\left(\mathbb{R}^{d}\right)\right), \mathbf{z} \in T\left(\mathbb{R}^{d}\right)$. In particular $\left\{e_{w}:|w| \leq N\right\}$ yields an orthonormal basis in the level- $N$ truncated space $T^{N}\left(\mathbb{R}^{d}\right)$, with inner product $\langle\cdot, \cdot\rangle=\langle\cdot, \cdot\rangle_{N}$. We now introduce a natural notion of differential equation associated to sgrms and a class of vector fields $\left(f_{1}, \ldots, f_{d}\right)$.
Definition 2.18. Let $\mathbf{X}$ be $N$-sgrm or a good sgrm such that $\dot{\mathbf{X}}_{s, s} \in \mathcal{L}^{N}\left(\mathbb{R}^{d}\right)$. We say that a smooth path $Y:[0, T] \rightarrow \mathbb{R}^{e}$ is the solution of a differential equation driven by $\mathbf{X}$ with vector fields $\left(f_{1}, \ldots, f_{d}\right) \in\left(\operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)\right)^{d}$ if it satisfies the differential equation

$$
\begin{equation*}
\dot{Y}_{s}=\left\langle f\left(Y_{s}\right), \dot{\mathbf{X}}_{s, s}\right\rangle_{N}=\sum_{|w| \leq N} f_{w}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s}, w\right\rangle, \tag{20}
\end{equation*}
$$

where $f: \mathbf{x} \mapsto f_{\mathbf{x}}$ is given in (17). We will refer to equation (20) with the shorthand notation

$$
\begin{equation*}
d Y=f(Y) d \mathbf{X} \tag{21}
\end{equation*}
$$

Remark 2.19. We remark that the equation (21) generalises the known notion of controlled ODE. Indeed, $\left\langle\dot{\mathbf{X}}_{s, s}, \mathbf{1}\right\rangle=0$ for any $N$-sgrm or a good sgrm and if $\mathbf{X}$ is the minimal $N$ extension of a smooth path $X=\sum_{i=1}^{d} X^{i} e_{i}$, obtained by classical iterated integration in Example 2.12, then Definition 2.18 collapses to the usual definition of controlled equation

$$
\dot{Y}=\sum_{i=1}^{d} f_{i}(Y) \dot{X}^{i}
$$

thereby obtaining a consistent definition. However, the richer structure of smooth geometric rough paths in the examples 2.13 extends this framework.

Given a generic initial condition $Y_{0} \in \mathbb{R}^{e}$ and vector fields $\left(f_{1}, \ldots, f_{d}\right)$ on $\mathbb{R}^{e}$, it follows from standard properties on classical differential equations that there exists a unique solution with initial condition $Y_{0}$. For a $N$-sgrp $\mathbf{X}$, the notion of solution of differential equation driven by a sgrp is furthermore consistent with the notion of rough differential equation solution à la Davie [Dav08] when the driver $\mathbf{X}$ is interpreted as a $1 / N$-Hölder weakly geometric rough path.

Proposition 2.20. A path $Y:[0, T] \rightarrow \mathbb{R}^{e}$ is a solution of a differential equation driven by a $N-s g r p \mathbf{X}$ with vector fields $\left(f_{1}, \ldots, f_{d}\right)$ if and only if for all $s, t \in[0, T]$

$$
\begin{equation*}
Y_{t}-Y_{s}=\sum_{1 \leq|w| \leq N} f_{w}\left(Y_{s}\right)\left\langle\mathbf{X}_{s, t}, w\right\rangle+r_{s, t} \tag{22}
\end{equation*}
$$

where $\mathbf{X}_{s, t}$ is the geometric rough model associated to $\mathbf{X}_{t}$ and $r$ is a remainder such that $r_{s, t}=o(|t-s|)$ as $t \rightarrow s$.

Proof. Supposing the property (22), we first observe, since $f_{1}=0$ and $\left\langle\mathbf{X}_{s, s}, w\right\rangle=0$ for any non-empty word $w$,

$$
\lim _{t \downarrow s} \frac{Y_{t}-Y_{s}}{t-s}=\lim _{t \downarrow s} \sum_{1 \leq|w| \leq N} f_{w}\left(Y_{s}\right)\left\langle\mathbf{X}_{s}^{-1} \otimes_{N} \frac{\mathbf{X}_{t}-\mathbf{X}_{s}}{t-s}, w\right\rangle=\sum_{1 \leq|w| \leq N} f_{w}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s}, w\right\rangle
$$

We thus conclude that $Y$ is differentiable and obtain (20). (With $f$ and $s \mapsto \dot{\mathbf{X}}_{s, s}$ smooth, it is then clear that $Y$ is not only differentiable but in fact smooth.) Conversely, supposing that $Y$ is a solution of (20), we apply Taylor's formula to $Y$ and $t \rightarrow\left\langle\mathbf{X}_{s, t}, w\right\rangle$ for any $|w| \leq N$ obtaining for all $s, t \in[0, T]$.

$$
Y_{t}-Y_{s}=\sum_{1 \leq|w| \leq N} f_{w}\left(Y_{s}\right)\left\langle\dot{X}_{s, s}, w\right\rangle(t-s)+r_{s, t}^{\prime}=\sum_{1 \leq|w| \leq N} f_{w}\left(Y_{s}\right)\left\langle X_{s, t}, w\right\rangle+r_{s,}^{\prime \prime}
$$

for some $r_{s, t}^{\prime}, r_{s, t}^{\prime \prime}$ satisfying both $r_{s, t}^{\prime}, r_{s, t}^{\prime \prime}=o(|t-s|)$ as $t \rightarrow s$.
Using the properties of the diagonal derivative, we can restate the identity (20) with respect to the Lie algebra $\mathcal{L}^{N}\left(\mathbb{R}^{d}\right)$.

Lemma 2.21. Let $\mathbf{X}$ be $N$-sgrm or a good sgrm such that $\dot{\mathbf{X}}_{s, s} \in \mathcal{L}^{N}\left(\mathbb{R}^{d}\right)$ and consider $\mathfrak{B}^{N}$ an orthonormal basis for $\mathcal{L}^{N}\left(\mathbb{R}^{d}\right) \subset T^{N}\left(\mathbb{R}^{d}\right)$. Then we have the equivalence

$$
d Y=f(Y) d \mathbf{X} \quad \text { iff } \quad \dot{Y}_{s}=\sum_{\mathfrak{u} \in \mathfrak{B}^{N}} f_{\mathfrak{u}}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s}, \mathfrak{u}\right\rangle
$$

Proof. Complete $\mathfrak{B}=\mathfrak{B}^{N}$ to an orthonormal basis $\overline{\mathfrak{B}}=\mathfrak{B} \cup \mathfrak{B}^{\perp}$ of $T^{N}\left(\mathbb{R}^{d}\right)$. In that basis

$$
\dot{Y}_{s}=\left\langle f\left(Y_{s}\right), \dot{\mathbf{X}}_{s, s}\right\rangle_{N}=\sum_{\mathbf{x} \in \overline{\mathfrak{B}}} f_{\mathbf{x}}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s}, \mathbf{x}\right\rangle .
$$

We now observe that $\dot{\mathbf{X}}_{s, s} \in \mathcal{L}^{N}\left(\mathbb{R}^{d}\right)$, thanks to Proposition 2.6. Consequently, there is no contribution from any $\mathbf{x} \in \mathfrak{B}^{\perp} \subset \mathcal{L}^{N}\left(\mathbb{R}^{d}\right)^{\perp}$.

Example 2.22. In case $N=2$ a natural orthonormal basis for $\mathcal{L}^{2}\left(\mathbb{R}^{d}\right)$ is given by

$$
\left\{i, \frac{1}{\sqrt{2}}[i, j]: i, j \in\{1, \cdots, d\}, i<j\right\}
$$

and the previous lemma asserts that $d Y=f(Y) d \mathbf{X}$ is equivalent to the following higher order controlled equation

$$
\begin{aligned}
\dot{Y}_{s} & =\sum_{i=1}^{d} f_{i}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s}, i\right\rangle+\sum_{i<j} \frac{1}{2} f_{[i, j]}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s},[i, j]\right\rangle \\
& =\sum_{i=1}^{d} f_{i}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s}, i\right\rangle+\sum_{i<j} \frac{1}{2}\left[f_{i}, f_{j}\right]\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s},[i, j]\right\rangle .
\end{aligned}
$$

This also follows directly by applying Remark 2.7 to the equation (20) with $N=2$.

Remark 2.23. Even if a differential equation driven by a $N$-sgrm can be viewed as a Davie solution, we note however that Lemma 2.21 is a particular property of differential equations driven by $N$-smooth geometric rough paths which does not hold for RDEs driven by a generic $1 / N$-Hölder weakly geometric rough path.

Indeed, working with one-dimensional $1 / 2$-Hölder weakly geometric rough path ${ }^{3}$ and a generic vector field $f \in \operatorname{Vect}^{\infty}(\mathbb{R})$ one as the trivial identities

$$
\begin{aligned}
& \sum_{1 \leq|w| \leq 2} f_{w}\left(Y_{s}\right)\left\langle\mathbf{X}_{s, t}, w\right\rangle=f\left(Y_{s}\right)\left(X_{t}-X_{s}\right)+\left(f \cdot f^{\prime}\right)\left(Y_{s}\right) \frac{\left(X_{t}-X_{s}\right)^{2}}{2}, \\
& \sum_{\mathfrak{u} \in \mathfrak{B}^{2}} f_{\mathfrak{u}}\left(Y_{s}\right)\left\langle\mathbf{X}_{s, t}, \mathfrak{u}\right\rangle=f\left(Y_{s}\right)\left(X_{t}-X_{s}\right)
\end{aligned}
$$

Therefore there is no $r_{s, t}=o(|t-s|)$ such that

$$
\sum_{|w| \leq 2} f_{w}\left(Y_{s}\right)\left\langle\mathbf{X}_{s, t}, w\right\rangle=\sum_{\mathfrak{u} \in \mathfrak{B}^{2}} f_{\mathfrak{u}}\left(Y_{s}\right)\left\langle\mathbf{X}_{s, t}, \mathfrak{u}\right\rangle+r_{s, t} .
$$

### 2.4 Algebraic renormalization of SGRP

Consider a collection of Lie series $v=\left(v_{1}, \ldots, v_{d}\right) \subset \mathcal{L}\left(\left(\mathbb{R}^{d}\right)\right)$, and let $T_{v}$ be the $\otimes$ endomorphism on $T\left(\left(\mathbb{R}^{d}\right)\right)$, obtained by extending the translation map $e_{i} \mapsto e_{i}+v_{i}$. We call $T_{v}$ the translation map. In case $v_{i}$ are all Lie polynomials, we remark we that $T_{v}$ maps $T^{N}\left(\mathbb{R}^{d}\right)$ to $T^{M}\left(\mathbb{R}^{d}\right)$ with $M=N \cdot N^{\prime}$ where $N^{\prime}$ denotes the smallest integer such that $v_{i} \in \mathcal{L}^{N^{\prime}}\left(\mathbb{R}^{d}\right)$. By restriction to Lie series, $\mathcal{L}\left(\left(\mathbb{R}^{d}\right)\right) \subset T\left(\left(\mathbb{R}^{d}\right)\right)$, we can and will view $T_{v}$ also as Lie algebra endomorphism, still denoted by $T_{v}$. We first give a dynamic view on the higher-order translation of sgrp.

Theorem 2.24. (i) Let $v=\left(v_{1}, \ldots, v_{d}\right)$ be a collection of elements in $\mathcal{L}\left(\left(\mathbb{R}^{d}\right)\right)$. Given a sgrp $\mathbf{X}$ over $\mathbb{R}^{d}$, the unique solution to

$$
\begin{equation*}
\dot{\mathbf{Z}}_{t}=\mathbf{Z}_{t} \otimes\left(T_{v} \dot{\mathbf{X}}_{t, t}\right), \quad \mathbf{Z}_{0}=T_{v} \mathbf{X}_{0} \tag{23}
\end{equation*}
$$

takes values in $G\left(\mathbb{R}^{d}\right)$ and is again a sgrp, and we have the explicit form

$$
\begin{equation*}
\mathbf{Z}_{t}=T_{v}\left(\mathbf{X}_{t}\right), \quad \mathbf{Z}_{s, t}=T_{v}\left(\mathbf{X}_{s, t}\right) \tag{24}
\end{equation*}
$$

Starting with a sgrm $\mathbf{X}$ over $\mathbb{R}^{d}$, the above applies to $t \mapsto \mathbf{X}_{0, t}$ and we obtain a sgrm $\mathbf{Z}$ with explicit form given by

$$
\mathbf{Z}_{s, t}=T_{v}\left(\mathbf{X}_{s, t}\right)
$$

(ii) Let now $v=\left(v_{1}, \ldots, v_{d}\right)$ be a collection of elements in $\mathcal{L}\left(\mathbb{R}^{d}\right)$ and $\mathbf{X}$ be a good sgrm over $\mathbb{R}^{d}$. Then $\mathbf{Z}$ as constructed in (i) is also a good sgrm. More specifically, let $N^{\prime}$

[^2]be the smallest integer such that $v_{i} \in \mathcal{L}^{N^{\prime}}\left(\mathbb{R}^{d}\right)$ for all $i$ and assume $\mathbf{X}$ is the minimal extension of some $N$-sgrm $\mathbf{Y}$ over $\mathbb{R}^{d}$. Let $M=N \cdot N^{\prime}$. The unique solution to
$$
\dot{\mathbf{W}}_{t}=\mathbf{W}_{t} \otimes_{M}\left(T_{v} \dot{\mathbf{Y}}_{t, t}\right), \quad \mathbf{W}_{0}=1,
$$
defines a $M$-sgrp, given by $\mathbf{W}_{s, t}=\mathbf{W}_{s}^{-1} \otimes_{M} \mathbf{W}_{t}$, which we call $\mathcal{T}_{v}[\mathbf{Y}]$. Moreover, we have the explicit form
\[

$$
\begin{equation*}
\mathcal{T}_{v}[\mathbf{Y}]_{s, t}=T_{v}^{M}\left(\mathbf{Y}_{s, t}^{M}\right)=\operatorname{proj}_{M} T_{v}\left(\mathbf{X}_{s, t}\right) \tag{25}
\end{equation*}
$$

\]

with algebra endomorphism $T_{v}^{M}:=\operatorname{proj}_{M} T_{v} \mathfrak{i}^{M}$ of $\left(T^{M}\left(\mathbb{R}^{d}\right), \otimes_{M}\right)$, using the (linear) embedding $\mathfrak{i}^{M}: T^{M}\left(\mathbb{R}^{d}\right) \rightarrow T\left(\left(\mathbb{R}^{d}\right)\right)$, and $\mathbf{Y}^{M}=\operatorname{MinExt}^{M}(\mathbf{Y})$.

Remark 2.25. Note that the renormalization map $\mathcal{T}_{v}$ for an $N$-sgrm is neither a linear nor a pointwise map, in contrast to $\mathbf{X} \rightarrow T_{v} \mathbf{X}$ for sgrps or sgrms.

Proof. (i) Fix a $\operatorname{sgrp} \mathbf{X}$ and write

$$
\dot{\mathbf{X}}_{t}=\mathbf{X}_{t} \otimes\left(\mathbf{X}_{t}^{-1} \otimes \dot{\mathbf{X}}_{t}\right)=\mathbf{X}_{t} \otimes \dot{\mathbf{X}}_{t, t}
$$

Since $T_{v}$ commutes with derivation and is an algebra morphism, then $\mathbf{Z}_{t}=T_{v}\left(\mathbf{X}_{t}\right)$ clearly satisfies the differential equation (23). The algebraic properties of $T_{v}$ together with Proposition 2.6 imply that $T_{v}\left(\mathbf{X}_{t}\right)$ belongs to $G\left(\mathbb{R}^{d}\right)$ and $T_{v}\left(\dot{\mathbf{X}}_{t, t}\right)$ belongs to $\mathcal{L}\left(\left(\mathbb{R}^{d}\right)\right)$ respectively. Therefore $T_{v}\left(\mathbf{X}_{t}\right)$ must coincide with the Cartan development of $T_{v}\left(\mathbf{X}_{t, t}\right)$, thereby yielding (24). Similar results hold with a sgrm.
(ii) Fix a good sgrm and apply the part (i) to the path $t \mapsto \mathbf{X}_{0, t}$. Since $v$ contains only Lie polynomials the map $T_{v}$ becomes an algebra morphism $T_{v}: T(\mathbb{R}) \rightarrow T(\mathbb{R})$. Using Lemma 2.11 and this last property one has $T_{v}\left(\dot{\mathbf{X}}_{t, t}\right) \in \mathcal{L}\left(\mathbb{R}^{d}\right)$, which implies that $T_{v}\left(\mathbf{X}_{s, t}\right)$ is a good sgrm. In the specific case of a collections of elements belonging to $\mathcal{L}^{N^{\prime}}\left(\mathbb{R}^{d}\right)$, we consider $\mathbf{Y}^{M}=\operatorname{MinExt}{ }^{M}(\mathbf{Y})$. The path $t \mapsto \mathbf{Y}_{0, t}^{M}=\mathbf{Z}_{t}$ will satisfy the following equation

$$
\begin{equation*}
\dot{\mathbf{Z}}_{t}=\mathbf{Z}_{t} \otimes_{M} \dot{\mathbf{Y}}_{t, t}^{M}, \quad \mathbf{Z}_{0}=\mathbf{1}, \tag{26}
\end{equation*}
$$

Using stardard properties of tensors, $T_{v}^{M}$ is an algebra endomorphism of $\left(T^{M}\left(\mathbb{R}^{d}\right), \otimes_{M}\right)$. Indeed for any $x, y \in T^{M}\left(\mathbb{R}^{d}\right)$

$$
\begin{aligned}
\left(T_{v}^{M} x\right) \otimes_{M}\left(T_{v}^{M} y\right) & =\operatorname{proj}_{M}\left(\left(T_{v}^{M} x\right) \otimes\left(T_{v}^{M} y\right)\right)=\operatorname{proj}_{M}\left(\left(T_{v} x\right) \otimes\left(T_{v} y\right)\right) \\
& =\operatorname{proj}_{M} T_{v}(x \otimes y)=\operatorname{proj}_{M} T_{v}{ }^{M} \operatorname{proj}_{M}(x \otimes y)=T_{v}^{M}\left(x \otimes_{M} y\right)
\end{aligned}
$$

where we used in the fourth equality the fact that $T_{v} z \in J^{>M}$ for all $z \in J^{>M}$ ( $T_{v}$ is not lowering the grade). Applying $T_{v}^{M}$ to both sides of equation (26), we obtain that $T_{v}^{M}\left(\mathbf{Z}_{t}\right)$ coincides with the Cartan development in $G^{M}\left(\mathbb{R}^{d}\right)$ of $T_{v}^{M} \dot{\mathbf{Y}}^{M}$. By uniqueness of this we can express $\mathcal{T}_{v}[\mathbf{Y}]_{s, t}$ as $T_{v}^{M}\left(\mathbf{Z}_{s}\right)^{-1} \otimes_{M} T_{v}^{M}\left(\mathbf{Z}_{t}\right)=T_{v}^{M}\left(\mathbf{Y}_{s, t}^{M}\right)$, obtaining (25). The identity $T_{v}^{M}\left(\mathbf{Y}_{s, t}^{M}\right)=\operatorname{proj}_{M} T_{v}\left(\mathbf{X}_{s, t}\right)$ follows trivially.

Combining the properties of translation operators $T_{v}$ with the differential equation (21), we can describe the effect of translation on smooth geometric rough paths in the same way as BCFP19. Given $v=\left(v_{1}, \ldots, v_{d}\right)$ a collection of elements in $\mathcal{L}\left(\mathbb{R}^{d}\right)$ and
$\left(f_{1}, \cdots, f_{d}\right) \in\left(\operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)\right)^{d}$ we consider the collection of vector fields $\left(f_{1}^{v}, \ldots, f_{d}^{v}\right) \in$ $\left(\operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)\right)^{d}$ given for each $i=1, \cdots, d$ by

$$
f_{i}^{v}=f_{i}+f_{v_{i}}
$$

where $f_{v_{i}}$ was given in (19). Starting from this collection of translated vector fields on $d$ directions, we can consider the linear map $f^{v}: T\left(\mathbb{R}^{d}\right) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$, extending the family $\left(f_{1}^{v}, \ldots, f_{d}^{v}\right)$ on any word like in (17). This operation allows to transfer the action of the translation at the level of differential equations.

Theorem 2.26. Let $\mathbf{X}$ be a good sgrm and $\mathbf{W}$ a $N$-sgrm. For any given collection $v=\left(v_{1}, \ldots, v_{d}\right)$ of elements in $\mathcal{L}\left(\mathbb{R}^{d}\right)$ a path $Y:[0, T] \rightarrow \mathbb{R}^{e}$ solves one of the equation

$$
d Y=f(Y) d\left(T_{v}(\mathbf{X})\right), \quad d Y=f(Y) d\left(\mathcal{T}_{v}[\mathbf{W}]\right) ;
$$

if and only if it solves respectively

$$
d Y=f^{v}(Y) d \mathbf{X}, \quad d Y=f^{v}(Y) d \mathbf{W} .
$$

Remark 2.27. The differential equations in the above statement are understood, in the sense of Definition 2.18, as equations driven by a $N$-sgrm $\mathbf{W}$, a $\left(N \cdot N^{\prime}\right)$-sgrm $\mathcal{T}_{v}[\mathbf{W}]$ (when the directions $v=\left(v_{1}, \ldots, v_{d}\right)$ are all contained in $\mathcal{L}^{N^{\prime}}\left(\mathbb{R}^{d}\right)$ ) and a good $\operatorname{sgrm} T_{v} \mathbf{X}$, respectively.

Proof. Writing a $\mathbf{X}$ as the minimal extension of a $M$-sgrm for some $M \in \mathbb{N}$, the result follows by showing only the equivalence for $\mathbf{W}$. We present here two possible proofs.
(First argument) Fix $s$ and $y=Y_{s}$ and $\mathfrak{w}:=\dot{\mathbf{W}}_{s, s} \in \mathcal{L}^{N}\left(\mathbb{R}^{d}\right)$ and $M=N \cdot N^{\prime}$, where $N^{\prime}$ is the minimal integer such that $v_{i} \in \mathcal{L}^{N^{\prime}}\left(\mathbb{R}^{d}\right)$ for all $i=1, \cdots, d$. By Theorem 2.24, the diagonal derivative of $\mathcal{T}_{v}[\mathbf{W}]$ at time $s$ is precisely $T_{v} \mathfrak{w} \in \mathcal{L}^{M}\left(\mathbb{R}^{d}\right)$ up to minimal extension. Using Lemma 2.21, the result follows once we check that

$$
\sum_{\mathfrak{u} \in \mathfrak{B}^{M}} f_{\mathfrak{u}}(y)\left\langle T_{v} \mathfrak{w}, \mathfrak{u}\right\rangle=\sum_{\mathfrak{v} \in \mathfrak{B}^{N}} f_{\mathfrak{v}}^{v}(y)\langle\mathfrak{w}, \mathfrak{v}\rangle,
$$

To prove this identity, we use an argument already contained in [BCFP19]. Let $f: \mathfrak{x} \mapsto f_{\mathfrak{x}}$ be the induced Lie algebra morphism from $\mathcal{L}^{M}\left(\mathbb{R}^{d}\right)$ into Vect ${ }^{\infty}\left(\mathbb{R}^{e}\right)$. Hence, whenever $f$ is smooth, the maps $f_{T_{v}}$ and $f^{v}$ are both Lie algebra morphisms from $\mathcal{L}^{M}\left(\mathbb{R}^{d}\right)$ into Vect ${ }^{\infty}\left(\mathbb{R}^{e}\right)$, which furthermore agree on the generators $e_{i}$. Thus $f_{T_{v} \mathfrak{v}}(z)=f_{\mathfrak{x}}^{v}(z)$ for any $z \in \mathbb{R}^{e}$ and any $\mathfrak{x} \in \mathcal{L}^{M}\left(\mathbb{R}^{d}\right)$ thanks to the universal property of $\mathcal{L}^{M}\left(\mathbb{R}^{d}\right)$ so

$$
\sum_{\mathfrak{u} \in \mathfrak{B}^{M}} f_{\mathfrak{u}}(y)\left\langle T_{v} \mathfrak{w}, \mathfrak{u}\right\rangle=f_{T_{v} \mathfrak{w}}(y)=f_{\mathfrak{w}}^{v}(y)=\sum_{\mathfrak{v} \in \mathfrak{B}^{N}} f_{\mathfrak{v}}^{v}(y)\langle\mathfrak{w}, \mathfrak{v}\rangle .
$$

(Second argument) The identity $f_{T_{v}}(z)=f^{v}(z)$ does not hold simply over $\mathcal{L}\left(\mathbb{R}^{d}\right)$ but over all $T\left(\mathbb{R}^{d}\right)$. For that, we first show that in fact for any $u \in \mathcal{L}\left(\mathbb{R}^{d}\right)$ and any $x \in T^{>0}\left(\mathbb{R}^{d}\right)$ we have $f_{u} \triangleright f_{x}=f_{u \otimes x}$. We do so completely analogously to [Pre21, Lemma 2.5.11], proceeding by induction over the length of $u$. For $u$ a letter, the statement holds by definition of $w \mapsto f_{w}$. Assume the statement holds for all $u$ of length $n$. Then, to check
it for all homogeneous Lie elements of word length $n+1$, since left bracketings span the free Lie algebra, it suffices to look at Lie elements of the form $[u, i]$. So,

$$
\begin{aligned}
f_{[u, i] \otimes x} & =f_{u \otimes i \otimes x}-f_{i \otimes u \otimes x}=f_{u} \triangleright\left(f_{i} \triangleright f_{x}\right)-f_{i} \triangleright\left(f_{u} \triangleright f_{x}\right) \\
& =\left[f_{u}, f_{i}\right]_{\triangleright} \triangleright f_{x}=f_{[u, i]} \triangleright f_{x},
\end{aligned}
$$

where for the third equality, we used the pre-Lie identity. Then, since $v_{i} \in \mathcal{L}\left(\mathbb{R}^{d}\right)$, indeed we get, again via induction over the length of words,

$$
f_{i w}^{v}=f_{i}^{v} \triangleright f_{w}^{v}=f_{T_{v} i} \triangleright f_{T_{v} w}=f_{v_{i}} \triangleright f_{T_{v} w}=f_{v_{i} \otimes T_{v}(w)}=f_{T_{v}(i w)} .
$$

We finally obtain the general identity by linearity. Thus for any $\mathbf{x} \in T^{N}\left(\mathbb{R}^{d}\right)$ we have

$$
\sum_{w}\langle\mathbf{x}, w\rangle f_{w}^{v}=f_{\mathbf{x}}^{v}=f_{T_{v} \mathbf{x}}=\sum_{w}\left\langle T_{v} \mathbf{x}, w\right\rangle f_{w} .
$$

Then, by Proposition 2.20 we have the desired equivalence of differential equations through the following equality, for any smooth path $Y:[0, T] \rightarrow \mathbb{R}^{e}$ and $s, t \in[0, T]$, using the notation $\mathbf{W}^{\infty}:=\operatorname{MinExt}(\mathbf{W})$ one has

$$
\begin{aligned}
& Y_{t}-Y_{s}=\sum_{1 \leq|w| \leq N}\left\langle\mathbf{W}_{s, t}, w\right\rangle f_{w}^{v}\left(Y_{s}\right)=\sum_{1 \leq|w|}\left\langle\operatorname{proj}_{N} \mathbf{W}_{s, t}^{\infty}, w\right\rangle f_{w}^{v}\left(Y_{s}\right) \\
& =\sum_{1 \leq|w|}\left\langle T_{v} \operatorname{proj}_{N} \mathbf{W}_{s, t}^{\infty}, w\right\rangle f_{w}\left(Y_{s}\right)=\sum_{1 \leq|w|}\left\langle\operatorname{proj}_{M} T_{v} \mathbf{W}_{s, t}^{\infty}, w\right\rangle f_{w}\left(Y_{s}\right)+R_{s, t} \\
& =\sum_{1 \leq|w| \leq M}\left\langle\left(\mathcal{T}_{v}[\mathbf{W}]\right)_{s, t}, w\right\rangle f_{w}\left(Y_{s}\right)+R_{s, t},
\end{aligned}
$$

with

$$
\begin{aligned}
R_{s, t} & =\sum_{1 \leq|w| \leq M}\left\langle T_{v} \operatorname{proj}_{N} \mathbf{W}_{s, t}^{\infty}-T_{v} \mathbf{W}_{s, t}^{\infty}, w\right\rangle f_{w}\left(Y_{s}\right) \\
& =-\sum_{1 \leq|w| \leq M}\left\langle\operatorname{proj}_{>N} \mathbf{W}_{s, t}^{\infty}, T_{v}^{*} w\right\rangle f_{w}\left(Y_{s}\right)=o(|t-s|),
\end{aligned}
$$

as $\left\langle\mathbf{W}_{s, t}^{\infty}, w\right\rangle=O\left(|t-s|^{2}\right)$ for all $w$ with $|w|>N$, since then $(s, t) \mapsto\left\langle\mathbf{W}_{s, t}^{\infty}, w\right\rangle$ is a smooth function on the compact domain $[0, T]^{2}$ with $\left\langle\mathbf{W}_{t, t}^{\infty}, w\right\rangle=0$ and $\left.\partial_{s}\right|_{s=t}\left\langle\mathbf{W}_{s, t}^{\infty}, w\right\rangle=$ $\left\langle\dot{\mathbf{W}}_{t, t}^{\infty}, w\right\rangle=0$.

Remark 2.28. The second argument in the above proof is valid for genuine (non-smooth) $\gamma$-Hölder weakly geometric rough paths, with $N=\lfloor 1 / \gamma\rfloor, Y:[0, T] \rightarrow \mathbb{R}^{e}$ an arbitrary continuous path and choosing $\operatorname{MinExt}(\mathbf{W})$ as the Lyons extension of $\mathbf{W}$, see Proposition 2.14. There, it again correctly identifies the respective rough differential equations via Davies' expansion. This argument first presented in Pre21, Section 2.5.2] fills a gap left in the proof of [BCFP19, Theorem 38] where it is misleadingly suggested that a Lie algebraic argument can be used. Indeed, as noted in Remark 2.23 there is no Davie-type definition of RDE solution where the sum is restricted to a Lie basis.

As final application of the section we combine Theorems 2.24 and 2.26 with the notion of minimal coupling and sum to obtain a "smooth geometric" renormalisation along a time component, see BCFP19, Thm. 38]. Consider a time-space path of the form

$$
\bar{X}_{t}=\left(t, X_{t}\right)=\left(X_{t}^{0}, X_{t}^{1}\right) \in \mathbb{R} \oplus \mathbb{R}^{d}=\mathbb{R}^{1+d}
$$

for some smooth path $X^{1}:[0, T] \rightarrow \mathbb{R}^{d}$ (the first component of canonical basis of $\mathbb{R}^{d+1}$ will be denoted by $\left.e_{0}\right)$. Assume also that there exists a good sgrm $\mathbf{X}^{1}:[0, T]^{2} \rightarrow G\left(\mathbb{R}^{d}\right)$ extending $X^{1}$. Then we can introduce the usual signature of time component $\mathbf{X}^{0}$ : $[0, T]^{2} \rightarrow G(\mathbb{R})$ defined by

$$
\mathbf{X}_{s, t}^{0}=\exp _{\otimes}\left((t-s) e_{0}\right)
$$

and the choice

$$
\overline{\mathbf{X}}:=\left(\mathbf{X}^{0}, \mathbf{X}^{1}\right)_{\min }
$$

constitutes a canonical way to extend $\bar{X}$. Choosing then a Lie polynomial $v_{0} \in \mathcal{L}\left(\mathbb{R}^{d+1}\right)$, we can also consider the translation map $T_{v_{0}}$ obtained by determined by the identity action on $e_{1}, \ldots, e_{d}$, and $e_{0} \mapsto e_{0}+v_{0}$. Then we can explicitly describe $T_{v_{0}} \overline{\mathbf{X}}$ and differential equations driven by that.
Corollary 2.29. Let $\mathbf{X}^{1}$ be a good sgrm and $v_{0} \in \mathcal{L}(A)$. Then the translation $t \mapsto \hat{T}_{v_{0}} \overline{\mathbf{X}}_{0, t}$ can be described as the sum $\mathbf{V}_{0} \boxplus \overline{\mathbf{X}}$, where $\mathbf{V}_{0}$ is given by

$$
\left(\mathbf{V}_{0}\right)_{s, t}=\exp _{\otimes}\left(v_{0}(t-s)\right) .
$$

Moreover, for any given family $\left(f_{\hat{0}}, f_{a}: a \in A\right)$ one has the equivalence

$$
d Y=\bar{f}(Y) d\left(\hat{T}_{u_{0}}(\overline{\mathbf{X}})\right) \quad \text { iff } \quad d Y=\left(f_{0}+\bar{f}_{\Phi_{H}^{*} u_{0}}\right)(Y) d t+f(Y) d \mathbf{X}^{1}
$$

where $\bar{f}: T\left(\mathbb{R}^{d+1}\right) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$ and $f: T\left(\mathbb{R}^{d+1}\right) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$ are defined like in (19) starting respectively from $\left(f_{0}, \ldots, f_{d}\right)$ and $\left(f_{1}, \ldots, f_{d}\right)$.
Proof. The proof follows by applying the results in the section. By construction of $\overline{\mathbf{X}}$ one has the following diagonal derivative $\dot{\overline{\mathbf{X}}}_{t, t}=e_{0}+\dot{\mathbf{X}}_{t, t}^{1}$. Then by construction of $T_{v_{0}}$ and Theorem 2.24, we have

$$
\begin{equation*}
\left.\frac{d}{d t}\left(T_{v_{0}}(\overline{\mathbf{X}})_{s, t}\right)\right|_{t=s}=T_{v_{0}}\left(e_{0}+\dot{\mathbf{X}}_{s, s}^{1}\right)=v_{0}+e_{0}+\dot{\mathbf{X}}_{s, s}^{1}=\left.\frac{d}{d t}\left(\mathbf{V}_{0} \boxplus \overline{\mathbf{X}}\right)_{s, t}\right|_{t=s} \tag{27}
\end{equation*}
$$

Since the diagonal derivative identifies uniquely $T_{v_{0}}(\overline{\mathbf{X}})$ we conclude. Passing to the differential equation, we remark from Definition 2.18 that a smooth path $Y:[0, T] \rightarrow \mathbb{R}^{e}$ is a solution of $d Y=\bar{f}(Y) d \overline{\mathbf{X}}$ if and only if

$$
d Y=f_{0}(Y) d t+f(Y) d \mathbf{X}^{1}
$$

Moreover, the map $\bar{f}^{v_{0}}$ satisfies trivially $\bar{f}_{0}^{v_{0}}=f_{0}+\bar{f}_{v_{0}}, \bar{f}_{i}^{v_{0}}=f_{i}$ for any $i=1, \cdots, d$. We conclude then by Theorem 2.26.

## 3 Smooth quasi-geometric rough paths

In the sequel, we extend the previous results and notions of smooth rough paths in the context of quasi-geometric rough paths, the natural generalisation of rough paths when we consider an underlying quasi-shuffle algebra, see Hof00. Quasi-geometric rough paths were first introduced in talks by David Kelly and properly studied in Bel20.

### 3.1 Quasi-shuffle algebras

We briefly review quasi-shuffle algebras in the version of Hof00. Their origin can be traced back to [Car72] and we also refer the reader to [FP20, MP18]. We shall use Hoffman's isomorphism in the context of stochastic integration ('Itô vs. Stratonovich') which was discussed in detail in EMPW15].

Let us start from a generic vector space $\mathbb{R}^{A}$ where $A$ is a locally finite (with respect to the weight function $\omega$ below) set called alphabet. To link this space with the previous section, we assume the inclusion $\{1 \cdots, d\} \subset A$, which induces a canonical inclusion $\mathbb{R}^{d} \subset \mathbb{R}^{A}$. Given an alphabet $A$, with a slight abuse of notation we will denote by $T(A)$ and $T((A))$ the spaces of tensor polynomials and tensor series over $\mathbb{R}^{A}$ respectively. To simplify the notation, in what follows we will also identify elements of $T((A))$ as words, as explained in the previous section i.e.

$$
e_{a_{1}} \otimes \cdots \otimes e_{a_{n}} \longleftrightarrow a_{1} \cdots a_{n}
$$

for any $a_{1}, \cdots, a_{n} \in A$. Moreover, we use the same symbol $\otimes$ to denote the tensor product operation of $T((A))$ and the "external" algebraic tensor product of two vector spaces.

The quasi-shuffle product relies on the existence of a commutative bracket, i.e. a map $\{\}:, \bar{A} \times \bar{A} \rightarrow \bar{A}$, where $\bar{A}=A \cup\{0\}$ such that

$$
\begin{equation*}
\{a, 0\}=\{0, a\}=0, \quad\{a, b\}=\{b, a\}, \quad\{a,\{b, c\}\}=\{\{a, b\}, c\} . \tag{28}
\end{equation*}
$$

We adopt the notation $\{a\}=a$, for $a \in A$, and more generally, for any word $w=a_{1} \cdots a_{n}$ we set $\left\{a_{1} \cdots a_{n}\right\}:=\left\{a_{1},\left\{\cdots\left\{a_{n-1}, a_{n}\right\}\right\} \cdots\right\}$ independently of the order of the letters in the word and the brackets. Together with the choice of a commutative bracket $\{$, we also fix a weight $\omega$ which is compatible with $\{$,$\} , i.e. \omega$ is a function $\omega: A \rightarrow \mathbb{N}^{*}$ satisfying

$$
\begin{equation*}
\omega(\{a b\})=\omega(a)+\omega(b) \tag{29}
\end{equation*}
$$

when $\{a b\} \neq 0$. Note that there could be several weights compatible with a given bracket and the definition of bracket does not need a weight.

Given a word $w=a_{1} \ldots a_{n}$ we denote by $|w|=n$ the length of $w$ and introduce its weighted length $\|w\|=\omega\left(a_{1}\right)+\cdots+\omega\left(a_{n}\right)$. Since $\omega(a) \geq 1$ for all $a \in A$ we have trivially $|w| \leq\|w\|$. The weighted length induces a grading on $T(A)$. To stress this property, we also use the notations $T_{\omega}(A)$ and $T_{\omega}((A))$ when we grade these vector spaces according to the weighted length. e.g.

$$
T_{\omega}(A)=\bigoplus_{n=0}^{\infty}\langle\{\text { words } w:\|w\|=n\}\rangle .
$$

Moreover, for any $N \in \mathbb{N}$ we define the truncated space $T_{\omega}^{N}(A)$ by taking words of weighted length at most $N$, we denote the projection on $T_{\omega}^{N}(A)$ by $\operatorname{proj}_{N, \omega}$. Using the weighted length, we can view $T_{\omega}^{N}(A)$ as a quotient algebra and introduce the corresponding truncated tensor product $\otimes_{N, \omega}$. The pairing $\langle$,$\rangle between T\left(\left(\mathbb{R}^{d}\right)\right)$ and $T\left(\mathbb{R}^{d}\right)$ generalises to the pairing $\langle\rangle:, T_{\omega}((A)) \times T_{\omega}(A) \rightarrow \mathbb{R}$ for any alphabet $A$. Its restriction $T(A) \times T(A) \subset T((A)) \times T(A)$ yields a scalar product on $T(A)$. We denote by $\langle\cdot, \cdot\rangle_{N}$ the
restriction to truncated spaces $T_{\omega}^{N}(A) \times T_{\omega}^{N}(A)$ ．As before，the set $\{w:\|w\| \leq N\}$ is an orthonormal basis of $T_{\omega}^{N}(A)$ ．Combining $\{$,$\} and the shuffle product，we introduce the$ quasi－shuffle product．

Definition 3．1．The quasi－shuffle product $\widehat{\mathbb{W}}$ is the unique bi－linear map in $T(A)$ satisfying the relation $\mathbf{1} \widehat{\amalg} v=v \widehat{\mathbf{1}}=v$ for any $v \in T(A)$ and the recursive identity

$$
\begin{equation*}
v a \widehat{\amalg} w b=(v \widehat{\amalg} w b) a+(v a \widehat{\varpi} w) b+(v \widehat{\varpi} w)\{a b\}, \tag{30}
\end{equation*}
$$

for any words $v, w \in T(A)$ and letters $a, b \in A$ ．
Example 3．2．If $A=\{1, \cdots, d\}$ and the commutative bracket is trivial，i．e．$\{a b\}=0$ for any $a, b \in A$ ，then the recursive relation（30）reduces to

$$
v a \widehat{山} w b=(v \widehat{\amalg} w b) a+(v a \widehat{\amalg} w) b .
$$

so that $\widehat{山}$ coincides with the standard shuffle product $ш$ in（4）．Observe that setting $\omega(i)=1$ for all $i \in A$ ，one has $\|w\|=|w|$ and $T_{\omega}(\{1 \cdots d\})$ is exactly $T\left(\mathbb{R}^{d}\right)$ but in general we can consider the shuffle algebra over a generic alphabet with a weight which is not identically 1 ，see e．g．in HK15］．

Example 3．3．For fixed integers $M, d \geq 1$ ，we define the alphabet $\mathbb{A}_{M}^{d}$ as

$$
\mathbb{A}_{M}^{d}=\left\{\alpha \in \mathbb{N}^{d} \backslash\{0\}: \sum_{i=1}^{d} \alpha_{i} \leq M\right\}
$$

as well as the weight $\omega(\alpha)=\sum_{i=1}^{d} \alpha_{i}$ ．We also define the bracket $\{,\}_{M}: \overline{\mathbb{A}_{M}^{d}} \times \overline{\mathbb{A}_{M}^{d}} \rightarrow \overline{\mathbb{A}_{M}^{d}}$ as the unique bilinear map satisfying the property

$$
\{\alpha, \beta\}_{M}:= \begin{cases}\alpha+\beta & \text { if } \alpha+\beta \in \mathbb{A}_{M}^{d},  \tag{31}\\ 0 & \text { otherwise },\end{cases}
$$

for any $\alpha, \beta \in \mathbb{A}_{M}^{d}$ ．The sum arising in this expression is the intrinsic sum between multi－indices．The set $\{1, \cdots, d\}$ embeds in $\mathbb{A}_{M}^{d}$ for any $M \geq 0$ by simply sending every $i \in\{1, \cdots, d\}$ to the $i$－th element of the canonical basis．Using this inclusion the letter $\{112\}$ is identified with the multi－index $(2,1,0, \ldots 0)$ and e have $\left\{i_{1} \cdots i_{n}\right\}=0$ for any $n>M$ and $i_{j} \in\{1, \cdots, d\}$ ．It follows from standard properties on multi－indices that $\{$,$\} and \omega$ satisfy properties（28）and（29）．The associated quasi－shuffle algebra $T_{\omega}\left(\mathbb{A}_{M}^{d}\right)$ is used to define the notion of quasi－geometric bracket extension in Bel20．We can also drop the finite index $M$ and consider the infinite alphabet $\mathbb{A}^{d}=\mathbb{N}^{d} \backslash\{0\}$ with the same weight．This alphabet is then isomorphic to the free semi－group generated over $d$ elements．The associated quasi－shuffle algebra was intensively used in［DET20］．

For any choice of $\{$,$\} the quasi－shuffle product \widehat{山}$ is a well－defined commutative product on $T(A)$ ，see Hof00．It further defines a product on the graded algebra $T_{\omega}(A)$ for a given compatible weight function $\omega: A \rightarrow \mathbb{N}^{*}$ ．The product w is also compatible with the grading on $T_{\omega}(A)$ ，giving rise to two algebra structures on the same graded vector space．

From a Hopf algebraic point of view, see Section 4 for the general framework, we can equip with $T_{\omega}(A)$ with the deconcatenation coproduct $\Delta: T_{\omega}(A) \rightarrow T_{\omega}(A) \otimes T_{\omega}(A)$ and the counit $\mathbf{1}^{*}: T_{\omega}(A) \rightarrow \mathbb{R}$ defined respectively by

$$
\begin{gathered}
\Delta\left(a_{i_{1}} \cdots a_{i_{n}}\right)=\mathbf{1} \otimes a_{i_{1}} \cdots a_{i_{n}}+\sum_{k=1}^{n} a_{i_{1}} \cdots a_{i_{k}} \otimes a_{i_{k+1}} \cdots a_{i_{n}}+a_{i_{1}} \cdots a_{i_{n}} \otimes \mathbf{1} \\
\mathbf{1}^{*}(w):= \begin{cases}1 & \text { if } w=\mathbf{1} \\
0 & \text { otherwise }\end{cases}
\end{gathered}
$$

Both the triples $\left(T_{\omega}(A), \widehat{\amalg}, \Delta\right)$ and $\left(T_{\omega}(A), \amalg, \Delta\right)$ are graded Hopf algebras, see Hof00. A fundamental result is the existence of an explicit isomorphism between these two Hopf algebras for any commutative bracket $\{$,$\} , see Hof00, Thm. 3.3].$

Theorem 3.4 (Hoffman Isomorphism). For any commutative bracket $\{$,$\} we define the$ Hoffman "exponential" and"logarithm" $\Phi_{H}, \Psi_{H}: T(A) \rightarrow T(A)$ on any word as

$$
\begin{equation*}
\Phi_{H}(w)=\sum_{I \in C(|w|)} \frac{1}{I!}\{w\}_{I}, \quad \Psi_{H}(w)=\sum_{I \in C(|w|)} \frac{(-1)^{|w|-|I|}}{I}\{w\}_{I}, \tag{32}
\end{equation*}
$$

where $C(|w|)$ is the set of compositions of order $|w|$ i.e. the multi-indices $I=\left(i_{1}, \cdots, i_{m}\right)$ such that $i_{1}+\cdots+i_{m}=|w|$ and

$$
I!=i_{1}!\cdots i_{m}!, \quad I=i_{1} \cdots i_{m}, \quad|I|=m
$$

Moreover, we denote by $\{w\}_{I}$ the word

$$
\{w\}_{I}:=\left\{w_{1} \cdots w_{i_{1}}\right\}\left\{w_{i_{1}+1} \cdots w_{i_{1}+i_{2}}\right\} \cdots\left\{w_{i_{1}+\cdots+i_{m-1}+1} \cdots w_{n}\right\} .
$$

The map $\Phi_{H}:\left(T_{\omega}(A), \amalg, \Delta\right) \rightarrow\left(T_{\omega}(A), \widehat{山}, \Delta\right)$ is an isomorphism of graded Hopf algebras such that $\Psi_{H}=\Phi_{H}^{-1}$.

With $\Phi_{H}$ and $\Psi_{H}$ as in (32), two explicit formulae were shown in Hof00 for the adjoint maps $\Phi_{H}^{*}, \Psi_{H}^{*}: T_{\omega}(A) \rightarrow T_{\omega}(A)$ with respect to scalar product $\langle$,$\rangle . Indeed, for$ any word $w=a_{1} \cdots a_{n}$ we have

$$
\begin{equation*}
\Phi_{H}^{*}(w)=\Phi_{H}^{*}\left(a_{1}\right) \cdots \Phi_{H}^{*}\left(a_{n}\right), \quad \Psi_{H}^{*}(w)=\Psi_{H}^{*}\left(a_{1}\right) \cdots \Psi_{H}^{*}\left(a_{n}\right) \tag{33}
\end{equation*}
$$

and for each letter $a \in A$, the following identity holds:

$$
\begin{equation*}
\Phi_{H}^{*}(a)=\sum_{n \geq 1} \sum_{\left\{a_{1} \cdots a_{n}\right\}=a} \frac{1}{n!} a_{1} \cdots a_{n}, \quad \Psi_{H}^{*}(a)=\sum_{n \geq 1} \sum_{\left\{a_{1} \cdots a_{n}\right\}=a} \frac{(-1)^{n-1}}{n} a_{1} \cdots a_{n}, \tag{34}
\end{equation*}
$$

where the underlying set of summation $\left\{a_{1} \cdots a_{n}\right\}=a$ involves all possible ways of writing the letter $a$ in terms of brackets of the word $a_{1} \cdots a_{n}$. By simply considering the graded dual of the shuffle and quasi-shuffle Hopf algebras (see section (4) in Theorem 3.4, the map $\Phi_{H}^{*}$ becomes an isomorphism $\Phi_{H}^{*}:\left(T_{\omega}(A), \otimes, \Delta_{\widetilde{山}}\right) \rightarrow\left(T_{\omega}(A), \otimes, \Delta_{\amalg}\right)$ where $\Delta_{\widetilde{\omega}}, \Delta_{\mathrm{\omega}}: T_{\omega}(A) \rightarrow T_{\omega}(A) \otimes T_{\omega}(A)$ are the dual coproducts associated to the shuffle and quasi-shuffle operation. To extend $\Phi_{H}^{*}$ and $\Psi_{H}^{*}$ to the whole set $T((A))$, we observe
that the concatenation product $\otimes$ easily extends to $T_{\omega}((A))$. However, the coproducts $\Delta_{\widehat{\omega}}$ and $\Delta_{\amalg}$ do not make sense on $T_{\omega}((A)) \otimes T_{\omega}((A))$, see e.g. Pre16, Thm.3.6] for a counterexample on $T\left(\left(\mathbb{R}^{1}\right)\right)$. They need to be extended to the completed tensor space, see Reu93, Sec. 1.4]

$$
T_{\omega}((A)) \bar{\otimes} T_{\omega}((A)):=\prod_{k, m=0}^{\infty}\langle\{\text { words } w:\|w\|=k\}\rangle \otimes\langle\{\text { words } w:\|w\|=m\rangle,
$$

leading to two topological graded coalgebras $\left(T_{\omega}((A)), \Delta_{\widetilde{山}}\right)$ and $\left(T_{\omega}((A)), \Delta_{Ш}\right)$.
Proposition 3.5. The maps $\Phi_{H}^{*}$ and $\Psi_{H}^{*}$ defined by (33) and (34) uniquely extend to two linear maps $\Phi_{H}^{*}, \Psi_{H}^{*}: T_{\omega}((A)) \rightarrow T_{\omega}((A))$, for which we shall use the same notation. The map $\Phi_{H}^{*}$ gives rise to a graded automorphism of the algebra $\left(T_{\omega}((A)), \otimes\right)$ as well as an isomorphism between the two topological graded coalgebras $\left(T_{\omega}((A)), \Delta_{\mathbb{U}}\right)$ and $\left(T_{\omega}((A)), \Delta_{\amalg}\right)$. In both cases one has $\Psi_{H}^{*}=\left(\Phi_{H}^{*}\right)^{-1}$.

Proof. It is sufficient to build the extension of $\Phi_{H}^{*}$. Combining properties (33) and (34), for any series $S=\sum_{w} \alpha_{w} w \in T_{\omega}((A))$ we define

$$
\Phi_{H}^{*}(S):=\sum_{w} \sum_{n_{1} \geq 1} \sum_{\left\{a_{1}^{1} \cdots a_{n_{1}}^{1}\right\}=w_{1}} \cdots \sum_{n_{m} \geq 1} \sum_{\left\{a_{1}^{m} \cdots a_{n_{m}}^{m}\right\}=w_{m}} \alpha_{w} \frac{1}{n_{1}!} \cdots \frac{1}{n_{m}!} a_{1}^{1} \cdots a_{n_{m}}^{m}
$$

where $w=w_{1} \cdots w_{m}$. Thanks to property (29) and the assumption $\omega(A) \subset \mathbb{N}^{*}$, the terms in the series $\Phi_{H}^{*}(S)$ are locally finite, i.e. for any word $u$ there is only a finite number of non-zero terms in the bracket $\left\langle\Phi_{H}^{*}(S), u\right\rangle$. It follows from property (33) that $\Phi_{H}^{*}$ extended to infinite series is also an algebra morphism. The other properties easily follow from the fact that $T_{\omega}(A)$ and $T_{\omega}(A) \otimes T_{\omega}(A)$ are dense subsets of $T_{\omega}((A))$ and $T_{\omega}((A)) \bar{\otimes} T_{\omega}((A))$.

The Hoffman isomorphism allows to define Lie groups and Lie algebras from the corresponding Lie groups and Lie algebras built from the corresponding shuffle structures.

In what follows, we use the notations $\mathcal{L}^{N}(A), G^{N}(A), \mathcal{L}(A), \mathcal{L}((A))$ and $G(A)=$ $\exp _{\otimes} \mathcal{L}((A))$ to denote respectively the $N$-step Lie polynomials, the free step- $N$ Lie group, the Lie polynomials and Lie series generated by $\mathbb{R}^{A}$ and $G(A)=\exp _{\otimes} \mathcal{L}((A))$, where $\exp _{\otimes}$ is given in (5). In case we fix a generic weight $\omega$ and $N \in \mathbb{N}$, we denote by $\mathcal{L}_{\omega}^{N}(A)$ and $G_{\omega}^{N}(A)$ the following quotients

$$
\mathcal{L}_{\omega}^{N}(A)=\mathcal{L}(A) / \mathcal{L}_{>N}^{\omega}(A), \quad G_{\omega}^{N}(A)=G(A) / G_{>N}^{\omega}(A)
$$

where $\mathcal{L}_{>N}^{\omega}$ and $G_{>N}^{\omega}(A)$ are respectively the Lie ideal and normal subgroup

$$
\mathcal{L}_{>N}^{\omega}(A):=\langle\{w:\|w\|>N\}\rangle \cap \mathcal{L}(A), \quad G_{>N}^{\omega}:=\langle\{w:\|w\|>N\}\rangle \cap G(A) .
$$

(These last two properties follow trivially from (29)). Moreover, by passing to the quotient one has the identification

$$
\begin{align*}
& G_{\omega}^{N}(A)= \\
& \left\{\mathbf{x} \in T_{\omega}^{N}(A):\langle\mathbf{x}, v ш w\rangle=\langle\mathbf{x}, v\rangle\langle\mathbf{x}, w\rangle \text { on words } w, v \text { s.t. }\|w\|+\|v\| \leq N\right\} \tag{35}
\end{align*}
$$

and the property $G_{\omega}^{N}(A)=\exp _{\otimes_{N, \omega}}\left(\mathcal{L}_{\omega}^{N}(A)\right)$ where $\exp _{\otimes_{N, \omega}}: T_{\omega}^{N}(A) \rightarrow T_{\omega}^{N}(A)$ is the exponential associated with the truncated product $\otimes_{N, \omega}$

$$
\begin{equation*}
\exp _{\otimes_{N, \omega}} v=\sum_{n \geq 0}^{N} \frac{v^{\otimes_{N, \omega}^{n}}}{n!} . \tag{36}
\end{equation*}
$$

thereby obtaining that $\mathcal{L}_{\omega}^{N}(A)$ is the Lie algebra of $G_{\omega}^{N}(A)$. Similar objects can be defined in the quasi-shuffle context via the map $\Psi_{H}^{*}$. The proof of the following corollary is a straightforward application of Proposition 3.5 combined with the usual properties of Lie series and the Group $G(A)$.

Corollary 3.6. (i) The set $\hat{G}(A):=\Psi_{H}^{*} G(A)=\left\{\Psi_{H}^{*} \beta, \beta \in G(A)\right\}$ with the operation $\otimes$ is a group which has the following description

$$
\begin{equation*}
\hat{G}(A)=\{\mathbf{x} \in T((A)):\langle\mathbf{x}, v \widehat{\varpi} w\rangle=\langle\mathbf{x}, v\rangle\langle\mathbf{x}, w\rangle \text { for all words } w, v\} . \tag{37}
\end{equation*}
$$

We call $\hat{G}(A)$ the group of quasi-shuffle characters.
(ii) The sets $\hat{\mathcal{L}}((A))=\Psi_{H}^{*} \mathcal{L}((A))$ and $\hat{\mathcal{L}}(A):=\Psi_{H}^{*} \mathcal{L}(A)$ with the commutator of $\otimes$ are two Lie algebras, which satisfy $\hat{\mathcal{L}}(A) \subset \hat{\mathcal{L}}((A))$ and $\hat{G}(A)=\exp _{\otimes} \mathcal{L}((A))$. We call them the quasi-shuffle Lie series and quasi-shuffle Lie polynomials respectively and they coincide with the free Lie algebra and Lie series generated by the set $\mathfrak{A}=\left\{\Psi_{H}^{*}(a): a \in A\right\}$.
(iii) The set $\hat{G}_{\omega}^{N}(A):=\Psi_{H}^{*} G_{\omega}^{N}(A)$ with the operation $\otimes_{N, \omega}$ is a Lie group whose Lie algebra is given by $\hat{\mathcal{L}}_{\omega}^{N}(A):=\Psi_{H}^{*} \mathcal{L}_{\omega}^{N}(A)$ and one has $\hat{G}_{\omega}^{N}(A)=\exp _{\otimes_{N, \omega}}\left(\hat{\mathcal{L}}_{\omega}^{N}(A)\right)$ and the identification

$$
\begin{align*}
& \hat{G}_{\omega}^{N}(A)= \\
& \left\{\mathbf{x} \in T_{\omega}^{N}(A):\langle\mathbf{x}, v \widehat{山} w\rangle=\langle\mathbf{x}, v\rangle\langle\mathbf{x}, w\rangle \text { on words } w, v \text { s.t. }\|w\|+\|v\| \leq N\right\} . \tag{38}
\end{align*}
$$

Example 3.7. Let us spell out the primitive elements of $\hat{\mathcal{L}}_{\omega}^{3}\left(\mathbb{A}_{2}^{d}\right)$. Starting from the definition of $\mathcal{L}_{\omega}^{3}\left(\mathbb{A}_{2}^{d}\right)$ and using the notations [,] for the Lie bracket, one has that $\mathcal{L}_{\omega}^{3}\left(\mathbb{A}_{2}^{d}\right)$ is generated as vector space by the following elements

$$
\{i,[i, j],\{i j\},[[i, j], k],[i,\{j k\}]\}
$$

where $i, j, k \in\{1, \cdots, d\}$. It follows from the definition of $\Psi_{H}^{*}$ that

$$
\Psi_{H}^{*}(i)=i, \quad \Psi_{H}^{*}(\{i j\})=\{i j\}-\frac{1}{2}(i j+i j) \mathbf{1}_{i \neq j}-\frac{1}{2} i i \mathbf{1}_{i=j} .
$$

We therefore obtain that $\hat{\mathcal{L}}_{\omega}^{3}\left(\mathbb{A}_{2}^{d}\right)$ is generated as vector space by

$$
\left\{i,[i, j], \Psi_{H}^{*}(\{i j\}),[[i, j], k],\left[i, \Psi_{H}^{*}(\{j k\})\right]\right\} .
$$

### 3.2 Smooth quasi-geometric rough paths

The notion and properties of smooth rough paths can be transposed to the quasi-shuffle context leading to equivalent concepts.

Definition 3.8. We call level- $N$ smooth quasi-geometric rough path over the alphabet $A$ with weight $\omega$ (in short: $N$-sqgrp) any non zero path $\mathbf{X}:[0, T] \rightarrow T_{\omega}^{N}(A)$ satisfying the following properties:
(a' i) for all times $t \in[0, T]$ and all words $v$ and $w$ such that $\|w\|+\|v\| \leq N$

$$
\begin{equation*}
\left\langle\mathbf{X}_{t}, v \widehat{山} w\right\rangle=\left\langle\mathbf{X}_{t}, v\right\rangle\left\langle\mathbf{X}_{t}, w\right\rangle . \tag{39}
\end{equation*}
$$

(a' ii) For every word of weighted length $\|w\| \leq N$, the map $t \mapsto\left\langle\mathbf{X}_{t}, w\right\rangle$ is smooth and we write $\dot{\mathbf{X}}_{t}$ for the derivative of $\mathbf{X}$.

We call level- $N$ smooth quasi-geometric rough model (in short: $N$-sqgrm) any map $\mathbf{X}:[0, T]^{2} \rightarrow T_{\omega}^{N}(A)$ such that the property (39) holds for all $\mathbf{X}_{s, t}$ and the properties (b.ii), (b.iii) in Definition 2.3 hold on a set of weighted words with the operation $\otimes_{N, \omega}$. By smooth quasi-geometric rough path (model) (in short: sqgrp and sqgrm) we mean a path (map) with values in the full space of tensor series $T_{\omega}((A))$ where (39) hold for any $w, v \in T_{\omega}((A))$ and relation (b.ii) hold with $\otimes$.

It follows immediately from Definition 3.8 that sqgrps and $N$-sqgrps are identified with smooth paths with values in the groups $\hat{G}(A)$ or $\hat{G}_{\omega}^{N}(A)$ and we can pass to the associated models by considering the increments with respect to $\otimes_{N, \omega}$. Similarly we can easily adapt the notion of extension of a sqgrp and the diagonal derivative of a sqgrm, which takes value in $\hat{\mathcal{L}}((A))$ or $\hat{\mathcal{L}}_{\omega}^{N}(A)$. Since the shuffle product is a specific case of the quasi-shuffle product, Definition 3.8 extends also the notions of $N$-sgrps and $N$-sgrms in a presence of a generic alphabet $A$ and weight $\omega$, we will call these objects weighted $N$-sgrps and weighted $N$-sgrms.

Thanks to the properties of the Hoffman isomorphism, we can easily relate smooth (resp. truncated) quasi-geometric rough paths and models with some corresponding weighted geometric objects by simply composing these objects with the appropriate version of the Hoffman isomorphism.

Theorem 3.9. Let $N \geq 0$ and $\mathbf{X}:[0, T] \rightarrow T_{\omega}^{N}(A)$. Then $\mathbf{X}$ is a $N$-sqgrp if and only if $\hat{\mathbf{X}}=\Phi_{H}^{*} \mathbf{X}$ is a weighted $N$-sgrp. Conversely, $\hat{\mathbf{X}}$ is a weighted $N$-sgrp if and only if $\mathbf{X}=\Psi_{H}^{*} \hat{\mathbf{X}}$ is a $N$-sqgrp. The same result applies to $N$-sqgrms, sqgrps and sqgrms.
Proof. The result follows easily from combining the properties of the maps $\Phi_{H}^{*}$ and $\Psi_{H}^{*}$ in Theorem 3.4 with the conditions in Definition 2.1 and Definition 3.8. E.g. starting from a $N$-sqgrp X one has

$$
\begin{gathered}
\left\langle\hat{\mathbf{X}}_{t}, v \amalg w\right\rangle=\left\langle\mathbf{X}_{t}, \Phi_{H}(v ш w)\right\rangle=\left\langle\mathbf{X}_{t}, \Phi_{H}(v) \widehat{山} \Phi_{H}(w)\right\rangle= \\
=\left\langle\mathbf{X}_{t}, \Phi_{H}(v)\right\rangle\left\langle\mathbf{X}_{t}, \Phi_{H}(w)\right\rangle=\left\langle\hat{\mathbf{X}}_{t}, v\right\rangle\left\langle\hat{\mathbf{X}}_{t}, w\right\rangle
\end{gathered}
$$

for all words with joint weighted length $\|w\|+\|v\| \leq N$. Moreover, restricting $\Phi_{H}^{*} \mathbf{X}_{t}$ to all words with joint weighted length $\|w\| \leq N$, it will be a finite linear combination of the functions $\left\langle\mathbf{X}_{t}, v\right\rangle$ with $\|v\| \leq N$, which are all smooth functions. The same considerations apply to $N$-sqgrms, sqgrps and sqgrms trivially. The converse case with $\Psi_{H}^{*}$ follows from the properties in Corollary 3.6.

Thanks to this one-to-one correspondence between quasi-geometric rough paths and geometric rough paths, we can import all the constructions of the previous section by simply checking that they preserve the maps $\Phi_{H}^{*}$ and $\Psi_{H}^{*}$. In particular, the notion of minimal extension for sgrms can easily be transposed to sqgrms.

Proposition 3.10. Given an $N$-sqgrm $\mathbf{Y}$ for some $N \in \mathbb{N}$, there exists exactly one sqgrm extension $\mathbf{X}$ of $\mathbf{Y}$ which is minimal in the sense that for all $s \in[0, T]$ one has

$$
\dot{\mathbf{X}}_{s, s} \in \hat{\mathcal{L}}_{\omega}^{N}(A)
$$

We call $q \operatorname{MinExt}(\mathbf{Y}):=\mathbf{X}$ the quasi-geometric minimal extension of $\mathbf{Y}$ and also $q \operatorname{MinExt}^{\mathrm{N}^{\prime}}(\mathrm{Y}):=\operatorname{proj}_{N^{\prime}, \omega} \mathbf{X}$, for $N^{\prime}>N$, the $N^{\prime}$-minimal extension of $\mathbf{Y}$. For a fixed interval $[s, t] \subset[0, T]$ it holds that $\mathbf{X}_{s, t}$ only depends on $\left\{\left.\mathbf{Y}\right|_{[u, v]}: s \leq u \leq v \leq t\right\}$ and we introduce the quasi-signature of $\mathbf{Y}$ on $[s, t]$ by $q \operatorname{Sig}\left(\left.\mathbf{Y}\right|_{[s, t]}\right):=\mathbf{X}_{s, t} \in \hat{G}(A)$. Moreover, we have the identities

$$
\begin{equation*}
q \operatorname{MinExt}(\mathbf{Y})=\Psi_{H}^{*} \operatorname{MinExt}\left(\Phi_{H}^{*} \mathbf{Y}\right), \quad q \operatorname{Sig}\left(\left.\mathbf{Y}\right|_{[s, t]}\right)=\Psi_{H}^{*} \operatorname{Sig}\left(\left.\Phi_{H}^{*} \mathbf{Y}\right|_{[s, t]}\right) \tag{40}
\end{equation*}
$$

A sqgrm $\mathbf{X}$ is called $a$ good sqgrm if it satisfies $\mathbf{X}=q \operatorname{MinExt}(\mathbf{Y})$ for some $N$-sqgrm Y.

Remark 3.11. The existence and uniqueness of a minimal extension for sqgrg hold on general grounds thanks to properties of Hopf algebras, see Theorem 4.2, However, Hoffman isomorphism gives us in (40) an explicit solution of the minimal extension in terms of the minimal extension related to a geometric rough path.

Proof. Thanks to Theorem [3.9, the map $\hat{\mathbf{Y}}=\Phi_{H}^{*} \mathbf{Y}$ is a weighted $N$-sgrp. Since the proof of Theorem [2.8 applies to any alphabet $A$ and any weight on letters $\omega$, there exists a unique minimal extension $\hat{\mathbf{X}}$ of $\hat{\mathbf{Y}}$. Beyond the fact that the function $\mathbf{X}:=\Psi_{H}^{*} \hat{\mathbf{X}}$ is a quasi-geometric rough path, its diagonal derivative satisfies

$$
\dot{\mathbf{X}}_{s, s}=\Psi_{H}^{*} \dot{\hat{\mathbf{X}}}_{s, s}=\Psi_{H}^{*} \dot{\hat{\mathbf{Y}}}_{s, s}=\Psi_{H}^{*} \Phi_{H}^{*} \dot{\mathbf{Y}}_{s, s}=\dot{\mathbf{Y}}_{s, s}
$$

The uniqueness of $\hat{\mathbf{X}}$ follows from the uniqueness of the minimal extension in the geometric case and the isomorphism properties of $\Psi_{H}^{*}$ and $\Phi_{H}^{*}$. From this uniqueness we deduce also the identities (40) straightforwardly.

In the same way as for geometric rough paths, the notion of smooth quasi-geometric rough path is consistent with its equivalent $\gamma$-Hölder version as introduced in Bel20. The proof of the following Corolllary is left to the reader.

Corollary 3.12. Every $N$-sqgrm $\mathbf{X}$ is a $1 / N$-Hölder quasi-geometric rough path and its minimal extension $\mathbf{X}$ coincides with the lift of $\mathbf{X}$, as constructed in [Bel20, Prop. 3.9].

Remark 3.13. Even if in the literature there is no difference between $\gamma$-Hölder quasigeometric rough paths and weakly $\gamma$-Hölder quasi-geometric rough paths, we can easily adapt the techniques in [FV10] to introduce properly these concepts and to prove that $N$-sqgrm $\hat{\mathbf{X}}$ are weakly $1 / N$-Hölder quasi-geometric rough paths and they are dense in the set of $\gamma$-Hölder quasi-geometric rough paths for $1 / \gamma \in(N, N+1)$.

The existence of a minimal extension allows two define a vector space structure over sqgrms and weighted sgrps. Indeed given two smooth quasi-geometric rough models $\mathbf{X}, \mathbf{Y}:[0, T]^{2} \rightarrow \hat{G}(A)$ and $\lambda \in \mathbb{R}$, we define their canonical sum $\mathbf{X} \boxplus \mathbf{Y}$ and their scalar multiplication $\lambda \boxtimes \mathbf{X}$ like in Definition 2.15. Since the operation of taking the diagonal derivative of a sqgrm commutes with the linear maps $\Phi_{H}^{*}$ and $\Psi_{H}^{*}$, from Proposition 4.16 we deduce the following corollary.

Corollary 3.14. Let $\mathbf{X}, \mathbf{Y}$ be smooth quasi-geometric rough paths/models. Then the map $(s, t) \rightarrow \mathbf{X}_{s, t} \otimes \mathbf{Y}_{s, t}$ and $\mathbf{X} \boxplus \mathbf{Y}$ satisfy the same relations in (15) and (14) respectively. Moreover $\Phi_{H}^{*}$ is a linear isomorphism with inverse $\Psi_{H}^{*}$ between the vector space of sqgrms and weighted sgrms endowed with $\boxplus$ and $\boxtimes$ 。

Remark 3.15. The sum of smooth quasi-geometric rough paths/models can be used to define a notion of minimal coupling like in (16). In this case, the elements of the sum are smooth geometric rough paths/models $\mathbf{X}^{i}:[0, T] \rightarrow \hat{G}\left(A_{i}\right)$, where $A_{i}, i=1, \cdots, m$ form a partition of $A$ where each $A_{i}$ is closed under the commutative bracket $\{$,$\} .$

### 3.3 Differential equations driven by SQGRP

For renormalisation purposes in the context of differential equations, we introduce the notion of differential equation driven by a smooth quasi-geometric rough path. Since we replaced our initial directions on $\mathbb{R}^{d}$ with an alphabet $A$, we assume the existence of a wider collection of smooth vector fields $\left(f_{a}: a \in A\right) \subset \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$. Moreover, in the same way as $\{1, \cdots, d\} \subset A$ we further fix a family $\left(f_{1}, \cdots, f_{d}\right) \subset\left(f_{a}: a \in A\right)$. As before, we can apply the operation $\triangleright$ to the elements $\left(f_{a}: a \in A\right)$ obtaining again a linear map $f: T_{\omega}(A) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$ defined by (17). This map induces a morphism of Lie algebras $f: \mathcal{L}(A) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$, which is uniquely determined by his values on $A$. By means of the adjoint Hoffman exponential $\Phi_{H}^{*}$, we can uniquely define a map similar to $f$, which preserves the quasi-shuffle Lie polynomials.

Proposition 3.16. Given a family of vector fields $\left(f_{a}: a \in A\right) \subset \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$, there exists a map $\hat{f}: T_{\omega}(A) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$ whose restriction on $\hat{\mathcal{L}}(A)$ is the unique morphism of Lie algebras $\hat{f}: \hat{L}(A) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$ which satisfies $\hat{f}\left(\Psi_{H}^{*}(a)\right)=f_{a}$ for any $a \in A$. This map is explicitly given by $\hat{f}=f_{\Phi_{H}^{*}}$.

Proof. By definition of $\hat{\mathcal{L}}(A)$, The map $f_{\Phi_{H}^{*}}$ satisfies clearly the properties of the statement. Moreover, thanks to the free structure of $\hat{\mathcal{L}}(A)$ described in Corollary 3.6, the condition $\hat{f}\left(\Psi_{H}^{*}(a)\right)=f_{a}$ uniquely determines a morphism of Lie algebras $\hat{f}: \hat{\mathcal{L}}(A) \rightarrow$ $\operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$.

Using the explicit definitions of $f$ and $\Phi_{H}^{*}$, we can then describe $\hat{f}$ as the unique linear map satisfying the conditions

$$
\begin{equation*}
\hat{f}_{w}=\left(\hat{f}_{a_{1}} \triangleright\left(\ldots \triangleright\left(\hat{f}_{a_{n-1}} \triangleright \hat{f}_{a_{n}}\right) \ldots\right), \quad \hat{f}_{a}=\sum_{n \geq 1} \sum_{\left\{a_{1} \cdots a_{n}\right\}=a} \frac{1}{n!} f_{a_{1} \cdots a_{n}},\right. \tag{41}
\end{equation*}
$$

for any words $w=a_{1} \cdots a_{n}$ and $a \in A$. We are now ready to introduce the notion of differential equation in the quasi-geometric context.

Definition 3.17. Let $\mathbf{X}$ be $N$-sqgrm or a good sgrm such that $\dot{\mathbf{X}}_{s, s} \in \mathcal{L}_{\omega}^{N}(A)$. We say that a smooth path $Y:[0, T] \rightarrow \mathbb{R}^{e}$ is the solution of a differential equation driven by $\mathbf{X}$ and the vector fields $\left(f_{a}: a \in A\right)$ if it satisfies

$$
\begin{equation*}
\dot{Y}_{s}=\left\langle\hat{f}\left(Y_{s}\right), \dot{\mathbf{X}}_{s, s}\right\rangle_{N}=\sum_{\|w\| \leq N} \hat{f}_{w}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s}, w\right\rangle \tag{42}
\end{equation*}
$$

where $\hat{f}: \mathbf{x} \mapsto \hat{f}_{\mathbf{x}}$ is given in (41). We will refer to equation (42) as

$$
\begin{equation*}
d Y=\hat{f}(Y) d \mathbf{X} \tag{43}
\end{equation*}
$$

Remark 3.18. This definition leaves the open problem for future work how to canonically define a notion of differential equation, when only given $d$ vector fields $f_{1}, \ldots, f_{d}$, without the need for an arbitrary extension to $\left(f_{a}: a \in A\right)$, at best in a way that it is consistent with the geometric and branched $d$ dimensional setting.

Combining the results in Proposition 2.20, Lemma 2.21 and Proposition 3.5, we can summarise three equivalent characterisations of (43) in the quasi-geometric context.
Proposition 3.19. Given a path $Y:[0, T] \rightarrow \mathbb{R}^{e}$ a $N$-sqgrp $\mathbf{X}$ and a family of vector fields $\left(f_{a}: a \in A\right)$, one has the following equivalent conditions:
i) $Y$ solves $d Y=\hat{f}(Y) d \mathbf{X}$.
ii) $Y$ solves $d Y=f(Y) d \hat{\mathbf{X}}$, where $\hat{\mathbf{X}}=\Phi_{H}^{*} \mathbf{X}$ is the $N$-level weighted geometric rough path obtained via Theorem 3.9.
iii) For all $s, t \in[0, T]$

$$
\begin{equation*}
Y_{t}-Y_{s}=\sum_{1 \leq\|w\| \leq N} \hat{f}_{w}\left(Y_{s}\right)\left\langle\mathbf{X}_{s, t}, w\right\rangle+r_{s, t} \tag{44}
\end{equation*}
$$

where $\mathbf{X}$ is the smooth quasi-geometric rough model associated to $\mathbf{X}$ and $r$ is a remainder such that $r_{s, t}=o(|t-s|)$ as $t \rightarrow s$.
iv) For any $\hat{\mathfrak{B}}^{N}$ orthonormal basis of $\hat{\mathcal{L}}_{\omega}^{N}(A) \subset T_{\omega}^{N}(A)$ one has

$$
\dot{Y}_{t}=\sum_{\mathfrak{u} \in \hat{\mathfrak{B}}^{N}} \hat{f}_{\mathfrak{u}}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{t, t}, \mathfrak{u}\right\rangle
$$

The results i), ii) and iii) hold also if $\mathbf{X}$ is a good sqgrm.
Proof. The equivalence $i) \Leftrightarrow$ iii) follows in the same way as in Proposition 2.20 and the equivalence $i) \Leftrightarrow i v$ ) is a consequence of the property $\dot{\mathbf{X}}_{s, s} \in \hat{\mathcal{L}}_{\omega}^{N}(A)$. To prove the last equivalence $i) \Leftrightarrow i i$ ) it is sufficient to prove the identity

$$
\begin{equation*}
\sum_{\mathfrak{u} \in \hat{\mathfrak{B}}^{N}} \hat{f}_{\mathfrak{u}}(y)\langle\mathfrak{x}, \mathfrak{u}\rangle=\sum_{\mathfrak{v} \in \mathfrak{B}^{N}} f_{\mathfrak{v}}(y)\left\langle\Phi_{H}^{*} \mathfrak{x}, \mathfrak{v}\right\rangle \tag{45}
\end{equation*}
$$

for any $\mathfrak{x} \in \hat{\mathcal{L}}_{\hat{\omega}}^{N}(A), y \in \mathbb{R}^{e}$ and any orthonormal basis $\mathfrak{B}^{N}$ of $\mathcal{L}_{\omega}^{N}(A)$. However, by definition of $\hat{f}$ and using the property $\Phi_{H}^{*} \mathfrak{x} \in \mathcal{L}_{\omega}^{N}(A)$ one has

$$
\sum_{\mathfrak{u} \in \hat{\mathfrak{B}}^{N}} \hat{f}_{\mathfrak{u}}(y)\langle\mathfrak{x}, \mathfrak{u}\rangle=\hat{f}_{\mathfrak{y}}(y)=f_{\Phi_{H}^{*}(\mathfrak{x})}(y)=\sum_{\mathfrak{v} \in \mathfrak{B}^{N}} f_{\mathfrak{v}}(y)\left\langle\Phi_{H}^{*} \mathfrak{x}, \mathfrak{v}\right\rangle
$$

The case wit a smooth model follows straightforwardly.

Example 3.20. Let us check how some properties of Proposition 3.19 appear when we fix $A=\mathbb{A}_{2}^{d}$ from example 3.3, a sqgrm $\mathbf{X}$ over $A$ and a generic family of vector fields $\left(f_{1}, \cdots, f_{d}\right)$ extended to $A$ by setting $f_{\{i j\}} \equiv 0$. The differential equation

$$
d Y=f\left(Y_{t}\right) d \hat{\mathbf{X}}
$$

is equivalent to

$$
\begin{equation*}
\dot{Y}_{s}=\sum_{|w| \leq N} f_{w}\left(Y_{s}\right)\left\langle\dot{\hat{\mathbf{X}}}_{s, s}, w\right\rangle \tag{46}
\end{equation*}
$$

where $w$ is a word with values in the alphabet $\{1, \cdots, d\}$. Using the identity $\hat{\mathbf{X}}:=\Phi_{H}^{*} \mathbf{X}$, we can rewrite (46) as

$$
\dot{Y}_{s}=\sum_{|w| \leq N} f_{w}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s}, \Phi_{H} w\right\rangle
$$

It follows from the definition of $\Phi_{H}$ in (32) that it is possible to write down a general combinatorial formula for $\Phi_{H}(w)$, see [EMPW15, Prop. 4.10]

$$
\Phi(w)=\sum_{u \in\{w\}} \frac{1}{2^{|w|-|u|}} u
$$

where $\{w\}$ consists of the words we can construct from $w$ by successively replacing any neighbouring pairs $i j$ in $w$ by \{ij\}. Switching the sum on $w$ and $u$, equation (46) becomes

$$
\begin{equation*}
\dot{Y}_{s}=\sum_{|w| \leq N} \sum_{u \in\{w\}} \frac{1}{2|w|-|u|} f_{w}\left(Y_{s}\right)\left\langle\dot{\hat{\mathbf{X}}}_{s, s}, u\right\rangle=\sum_{\|u\| \leq N} \sum_{\{w\}_{p}=u} \frac{1}{2^{|w|-|u|}} f_{w}\left(Y_{s}\right)\left\langle\dot{\hat{\mathbf{X}}}_{s, s}, u\right\rangle \tag{47}
\end{equation*}
$$

where the sum over $\{w\}_{p}=u$ involves all the ways one can write a word $u$ as $\{w\}_{I}$ of a word $w$ whose letters are in the alphabet $\{1, \cdots, d\}$ and $I \in C(|w|)$. The element

$$
\sum_{\{w\}_{p}=u} \frac{1}{2^{|w|-|u|}} f_{w}
$$

then corresponds to the expression $\hat{f}_{u}$ for any word $u$.
Using the shorthand notations $\left\langle\dot{\mathbf{X}}_{s, s}, a\right\rangle=\left\langle\dot{\hat{\mathbf{X}}}_{s, s}, a\right\rangle=\dot{X}_{s}^{a}$ for any $a \in A$, the expressions (46) and (47) in case $N=2$ become respectively

$$
\dot{Y}_{s}=\sum_{i=1}^{d} f_{i}\left(Y_{s}\right) \dot{X}_{s}^{i}+\sum_{i, j=1}^{d} f_{i j}\left(Y_{s}\right)\left\langle\dot{\hat{X}}_{s, s}, i j\right\rangle
$$

and

$$
\begin{aligned}
& \dot{Y}_{s}=\sum_{i=1}^{d} \hat{f}_{i}\left(Y_{s}\right) \dot{X}_{s}^{i}+\sum_{i, j=1}^{d} \hat{f}_{i j}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s}, i j\right\rangle+\sum_{i \leq j}^{d} \hat{f}_{\{i j\}}\left(Y_{s}\right) \dot{X}_{s}^{\{i j\}} \\
& =\sum_{i=1}^{d} f_{i}\left(Y_{s}\right) \dot{X}_{s}^{i}+\sum_{i, j=1}^{d} f_{i j}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s}, i j\right\rangle+\frac{1}{2}\left(\sum_{i=1}^{d} f_{i i}\left(Y_{s}\right) \dot{X}_{s}^{\{i i\}}+\sum_{i<j}^{d}\left(f_{i j}+f_{j i}\right)\left(Y_{s}\right) \dot{X}_{s}^{\{i j\}}\right) \\
& =\sum_{i=1}^{d} f_{i}\left(Y_{s}\right) \dot{X}_{s}^{i}+\sum_{i, j=1}^{d} f_{i j}\left(Y_{s}\right)\left\langle\dot{\mathbf{X}}_{s, s}, i j\right\rangle+\frac{1}{2} \sum_{i, j=1}^{d} f_{i j}\left(Y_{s}\right) \dot{X}_{s}^{\{i j\}} .
\end{aligned}
$$

Interpreting $X^{i}$ and $X^{\{i j\}}$ as the components of a semimartingale and his quadratic variations, we obtain an algebraic version of the Itô-Stratonovich correction among semimartingales at level of SDEs, see EMPW15] for further applications.

### 3.4 Algebraic renormalisation of SQGRP

We now want to adapt the notion of translation discussed for geometric rough paths to the context of quasi-geometric rough paths. Adapting the same arguments in the section [2.4, for any family $\left(v_{a}: a \in A\right) \subset \mathcal{L}((A))$ we can easily define a translation map $T_{v}: T_{\omega}((A)) \rightarrow T_{\omega}((A))$ defined on any alphabet $A$. In the case of a family $\left(v_{a}: a \in\right.$ $A) \subset \mathcal{L}(A)$ it follows from the properties of any weight $\omega$ that $T_{v}$ sends $T_{\omega}^{N}(A)$ to $T_{\omega}^{M}(A)$ with $M=N \cdot N^{\prime}$, where $N^{\prime}$ is the smallest integer such that $v_{i} \in \mathcal{L}_{\omega}^{N^{\prime}}(A)$. Passing to the quasi- shuffle context, the natural direction to perform a translation must be done along the Lie algebra $\hat{\mathcal{L}}((A))$. To define a proper notion, Corollary 3.6 tells us the set $\hat{\mathcal{L}}(A)$ consists of Lie series obtained from the set $\mathfrak{A}=\left\{\Psi_{H}^{*}(a): a \in A\right\}$. Therefore, fixing a subset $u=\left(u_{a}: a \in A\right) \subset \hat{\mathcal{L}}((A))$, there is a unique morphism of Lie algebras $\hat{T}_{u}: \hat{\mathcal{L}}((A)) \rightarrow \hat{\mathcal{L}}((A))$ such that

$$
\hat{T}_{u}\left(\Psi_{H}^{*}(a)\right)=\Psi_{H}^{*}(a)+u_{a}
$$

Using the translation maps $T_{v}$ and the maps $\Psi_{H}^{*}, \Phi_{H}^{*}$ we see that $\hat{T}_{u}$ uniquely extends to an endomorphism of $T_{\omega}((A))$ with respect to the product $\otimes$.
Theorem 3.21. For any collection $u=\left(u_{a}: a \in A\right) \subset \hat{\mathcal{L}}((A))$ there exists a unique $\otimes$-endomorphism over $T_{\omega}((A))$ which extends $\hat{T}_{u}: \hat{\mathcal{L}}((A)) \rightarrow \hat{\mathcal{L}}((A))$. We call it the quasi-translation map and we denote it by the same symbol $\hat{T}_{u}$. In particular, one has the explicit relation

$$
\begin{equation*}
\hat{T}_{u}=\Psi_{H}^{*} T_{\hat{u}} \Phi_{H}^{*} \tag{48}
\end{equation*}
$$

where $\hat{u}$ is given by the family $\hat{u}=\left(\Phi_{H}^{*}\left(u_{a}\right): a \in A\right) \subset \mathcal{L}((A))$. Moreover, if $\left(u_{a}: a \in\right.$ $A) \subset \hat{\mathcal{L}}(A) \hat{T}_{u}$ sends $T_{\omega}^{N}(A)$ to $T_{\omega}^{M}(A)$ with $M=N \cdot N^{\prime}$, where $N^{\prime}$ is the smallest integer such that $v_{i} \in \hat{\mathcal{L}}_{\omega}^{N^{\prime}}(A)$.

Proof. Thanks to Proposition 3.5 and the properties of the translation map, the map $\Psi_{H}^{*} T_{\hat{u}} \Phi_{H}^{*}$ is a $\otimes$-endomorphism satisfying

$$
\Psi_{H}^{*} T_{\hat{u}} \Phi_{H}^{*}\left(\Psi_{H}^{*}(a)\right)=\Psi_{H}^{*} T_{\hat{u}}(a)=\Psi_{H}^{*}\left(a+\Phi_{H}^{*}\left(u_{a}\right)\right)=\Psi_{H}^{*}(a)+u_{a} .
$$

Moreover, from Corollary 3.6 we also deduce that $\Psi_{H}^{*} T_{\hat{u}} \Phi_{H}^{*}$ is a Lie endomorphism of $\hat{\mathcal{L}}((A))$. Therefore we conclude from the freeness of $\hat{\mathcal{L}}((A))$ that the definition (48) is a bona fide extension of $\hat{T}_{u}$. Concerning the uniqueness of $\hat{T}_{u}$, let $\Gamma: T_{\omega}((A)) \rightarrow T_{\omega}((A))$ be an $\otimes$-endomorphism extending $\hat{T}_{u}$. Considering the map $\Phi_{H}^{*} \Gamma \Psi_{H}^{*}$ we can easily check on any $a \in A$ the property

$$
\Phi_{H}^{*} \Gamma \Psi_{H}^{*}(a)=a+\Phi_{H}^{*}\left(u_{a}\right)
$$

Using the fundamental property of $T((A)), \Phi_{H}^{*} \Gamma \Psi_{H}^{*}$ must coincide with $T_{\hat{u}}$ and we obtain the uniqueness. The last properties of $\hat{T}_{u}$ follows for the fact that $\Phi_{H}^{*}$ and $\Psi_{H}^{*}$ are graded maps.

Remark 3.22. This procedure still leaves open the question as how to define an endomorphism $T_{u}: T_{\omega}((A)) \rightarrow T_{\omega}((A))$ starting from a mere family $u=\left(u_{1}, \ldots, u_{d}\right)$ such that $T_{u} i=i+u_{i}$ for $i=1, \ldots, d$. This further question is motivated by the fact that such translations coming from (only) $d$ choices of Lie elements exist in a canonical way in the geometric and branched case, see also [BCFP19]. It would therefore be desirable to have such a translation map which is consistent with both the geometric and the branched $d$-dimensional setting.

The quasi-translation maps now at hand can be applied to sqgrps/sqgrms and their truncated versions as in Theorem 2.24. Indeed for any $u=\left(u_{a}: a \in A\right) \subset \hat{\mathcal{L}}((A))$ and sqgrp/sqgrm $\mathbf{X}$ we can actually define the compositions $\hat{T}_{u}\left(\mathbf{X}_{t}\right)$ and $\hat{T}_{u}\left(\mathbf{X}_{s, t}\right)$. In addition, for any $u=\left(u_{a}: a \in A\right) \subset \hat{\mathcal{L}}(A)$ and $N$-sqgrm $\mathbf{Y}$ we define

$$
\begin{equation*}
\hat{\mathcal{T}}_{u}[\mathbf{Y}]_{s, t}:=\hat{T}_{u}^{M}\left(q \operatorname{MinExt}^{M}(\mathbf{Y})\right)=\operatorname{proj}_{M, \omega} \hat{T}_{u}\left(q \operatorname{MinExt}(\mathbf{Y})_{s, t}\right) \tag{49}
\end{equation*}
$$

where $M=N \cdot N^{\prime}$ with $N^{\prime}$ the smallest integer such that $u_{a} \in \hat{\mathcal{L}}_{\omega}^{N^{\prime}}(A)$ for all $a \in A$ and $\hat{T}_{v}^{M}:=\operatorname{proj}_{M, \omega} \hat{T}_{u} \mathfrak{i}^{M}$ with the embedding $\mathfrak{i}^{M}: T_{\omega}^{M}(A) \rightarrow T_{\omega}((A))$. These two operations have equivalently a dynamical reinterpretation like Theorem 2.24.

Proposition 3.23. Given a sqgrp (sqgrm) $\mathbf{X}$ over $A$ with weight $\omega$ and $u=\left(u_{a}: a \in\right.$ $A) \subset \hat{\mathcal{L}}((A))$, the composition $\hat{T}_{u}\left(\mathbf{X}_{t}\right)\left(\hat{T}_{u}\left(\mathbf{X}_{s, t}\right)\right)$ is again a sqgrp (sqgrm) which coincides (up to increments) with the solution of

$$
\begin{equation*}
\dot{\mathbf{Z}}_{t}=\mathbf{Z}_{t} \otimes\left(\hat{T}_{u} \dot{\mathbf{X}}_{t, t}\right), \quad \mathbf{Z}_{0}=\hat{T}_{u} \mathbf{X}_{0} \tag{50}
\end{equation*}
$$

The same result applies to good sqgrms when $u \subset \hat{\mathcal{L}}(A)$. In addition, for any $u=\left(u_{a}: a \in\right.$ $A) \subset \hat{\mathcal{L}}(A)$ and $N$-sqgrm $\mathbf{Y}$ the path $t \rightarrow \hat{\mathcal{T}}_{u}[\mathbf{Y}]_{0, t}$ is a sqgrp coinciding with the solution of

$$
\begin{equation*}
\dot{\mathbf{W}}_{t}=\mathbf{W}_{t} \otimes_{M, \omega}\left(\hat{T}_{u} \dot{\mathbf{Y}}_{t, t}\right), \quad \mathbf{W}_{0}=\mathbf{1} \tag{51}
\end{equation*}
$$

where $M \in \mathbb{N}$ is contained in the definition of $\hat{\mathcal{T}}_{u}[\mathbf{Y}]$ in (49).
Proof. Choosing the family $\hat{u}=\left(\Phi_{H}^{*}\left(u_{a}\right): a \in A\right)$, we can apply Theorem 2.24 to the weighted sgrp $(\operatorname{sgrm}) \hat{\mathbf{X}}=\Phi_{H}^{*} \mathbf{X}$ and $\hat{\mathbf{Y}}=\Phi_{H}^{*} \mathbf{Y}$. Then the paths associated to $T_{\hat{u}}(\hat{\mathbf{X}})$ and $\mathcal{T}_{\hat{u}}[\hat{\mathbf{Y}}]$ solve respectively the equations

$$
\begin{gathered}
\dot{\hat{\mathbf{Z}}}_{t}=\hat{\mathbf{Z}}_{t} \otimes\left(T_{\hat{u}} \dot{\hat{\mathbf{X}}}_{t, t}\right), \quad \hat{\mathbf{Z}}_{0}=T_{\hat{u}} \hat{\mathbf{X}}_{0} \\
\dot{\hat{\mathbf{W}}}_{t}=\hat{\mathbf{W}}_{t} \otimes_{M, \omega}\left(T_{\hat{u}} \dot{\hat{\mathbf{Y}}}_{t, t}\right), \quad \hat{\mathbf{W}}_{0}=\mathbf{1}
\end{gathered}
$$

Using the identity (48) and the properties of $\Phi_{H}^{*}, \Psi_{H}^{*}$ of preserving the weighted length, we deduce that $\hat{T}_{u}(\mathbf{X})=\Psi_{H}^{*} T_{\hat{u}}(\hat{\mathbf{X}})$ and $\hat{\mathcal{T}}_{u}[\mathbf{Y}]=\Psi_{H}^{*} \mathcal{T}_{\hat{u}}[\hat{\mathbf{Y}}]$. Looking at the equation of the paths $\Psi_{H}^{*} \hat{\mathbf{Z}}$ and $\Psi_{H}^{*} \hat{\mathbf{W}}$, we obtain the equations (50) and (51).

We conclude the section with a summary of the properties of translation operators $\hat{T}_{u}$ together with the differential equation (21). For any collection of Lie polynomials with values in $A$ given by $v=\left(v_{a}: a \in A\right) \subset \mathcal{L}(A)$ and any family of vector fields $\left(f_{a}: a \in A\right)$, we consider the collection of vector fields $\left(f_{a}^{v}: a \in A\right)$ defined on any $a \in A$ as

$$
f_{a}^{v}:=f_{a}+f_{v_{a}}
$$

where $f: \mathcal{L}(A) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$ is the Lie algebra morphism defined in (17) on a generic alphabet $A$. Extending $\left(f_{a}^{v}: a \in A\right)$ to all $T_{\omega}(A)$, we obtain a map $f^{v}: T_{\omega}(A) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$ which satisfies the identity $f^{v}=f_{T_{v}}$ over $\mathcal{L}(A)$, as it was already shown in the proof of Theorem [2.26, when $A=\{1, \cdots, d\}$. By replacing the directions of translations $v$ with $u=\left(u_{a}: a \in A\right) \subset \hat{L}(A)$ we can uniquely define a Lie algebra morphism $\hat{f}^{u}: \hat{\mathcal{L}}(A) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$ such that on any $a \in A$ one has as

$$
\hat{f}_{\Psi_{H}^{*}(a)}^{u}:=\hat{f}_{\Psi_{H}^{*}(a)}+\hat{f}_{u_{a}} .
$$

Extending $\hat{f}^{u}$ to all $T_{\omega}(A)$ and applying the definition of $\hat{f}$ with formula (48), we also obtain a map $\hat{f}^{u}: T_{\omega}(A) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$. Both maps $\hat{f}^{u}$ and $f^{v}$ allow to write the effect of translation at the level of differential equations.

Theorem 3.24. Let $\mathbf{X}$ be a good sqgrm and $\mathbf{W} a N$-sqgrm. For any given $u=\left\{u_{a}: a \in\right.$ $A\} \subset \hat{\mathcal{L}}(A)$ a path $Y:[0, T] \rightarrow \mathbb{R}^{e}$ solves one of the equation

$$
d Y=\hat{f}(Y) d\left(\hat{T}_{u}(\mathbf{X})\right), \quad d Y=\hat{f}(Y) d\left(\hat{\mathcal{T}}_{u}[\mathbf{W}]\right)
$$

if and only if it solves respectively

$$
\begin{equation*}
d Y=\hat{f}^{u}(Y) d \mathbf{X}, \quad d Y=\hat{f}^{u}(Y) d \mathbf{W} . \tag{52}
\end{equation*}
$$

Alternatively, by setting $\hat{u}=\left\{\Phi_{H}^{*}\left(u_{a}\right): a \in A\right\} \subset \mathcal{L}(A)$ and $\hat{\mathbf{X}}=\Phi_{H}^{*} \mathbf{X}, \hat{\mathbf{W}}=\Phi_{H}^{*} \mathbf{W}$ then $Y$ solves also equivalently

$$
\begin{equation*}
d Y=f^{\hat{u}}(Y) d \hat{\mathbf{X}}, \quad d Y=f^{\hat{u}}(Y) d \hat{\mathbf{W}} . \tag{53}
\end{equation*}
$$

Proof. Equivalences (52) and (53) follow by combining the proof of Theorem 2.26 and Proposition 3.19. The only thing to check is to describe the relations between the maps $\hat{f}^{u}$ and $f^{\hat{u}}$ and the translations $T_{\hat{u}}$ and $\hat{T}_{u}$. Indeed, it follows from the the definition of $\hat{T}_{u}$ and $\hat{f}^{u}$ that one has

$$
\hat{f}^{u}=\hat{f}_{\hat{T}_{u}}
$$

on any element $\Psi_{H}^{*}(a)$ and consequently over all $\hat{\mathcal{L}}(A)$. Following the first argument in the Theorem [2.26 we get the first equivalence (52). Using the explicit definition of $\hat{f}$ and identity (48), we deduce also the following equality on $\hat{\mathcal{L}}(A)$

$$
\hat{f}_{\hat{T}_{u}}=f_{\Phi_{H}^{*} \hat{T}_{u}}=f_{T_{\hat{u}} \Phi_{H}^{*}}=f_{\Phi_{H}^{*}}^{\hat{u}} .
$$

Applying the same argument as in the proof of (ii) in Proposition 3.19, we conclude.
We conclude the section by extending the time translation of Corollary 2.29 in the quasi-shuffle context. For these purposes, we suppose given a generic alphabet $A^{1}$ with commutative bracket $\{,\}_{1}$ and weight $\omega_{1}$. Then we simply add an extra time component as new letter $\hat{0}$ and we define the extended alphabet $\bar{A}=\{\hat{0}\} \cup A^{1}$ together with weight $\omega: \bar{A} \rightarrow \mathbb{N}^{*}$ defined by extending $\omega_{1}$ and putting $\omega(\hat{0})=1$ and the extended commutative bracket $\{$,$\} given for any a, b \in A^{1}$

$$
\{a, \hat{0}\}=\{\hat{0}, a\}=\{\hat{0}, \hat{0}\}=0, \quad\{a, b\}=\{a, b\} .
$$

Under these conditions the signature of the time component $\mathbf{X}^{0}:[0, T]^{2} \rightarrow G(\mathbb{R})$

$$
\mathbf{X}_{s, t}^{0}:=\exp _{\otimes}((t-s) \hat{0})
$$

can be easily embedded in $\hat{G}(\bar{A})$ and for any good sqgrm $\mathbf{X}^{1}:[0, T]^{2} \rightarrow \hat{G}\left(A_{1}\right)$ we can introduce again

$$
\overline{\mathbf{X}}:=\left(\mathbf{X}^{0}, \mathbf{X}^{1}\right)_{\min } .
$$

(see also Remark (3.15). Since $\Psi_{H}^{*}(\hat{0})=\hat{0}$, for any given a $u_{0} \in \hat{\mathcal{L}}(\bar{A})$, we choose the translation $\hat{T}_{u_{0}}: T(A) \rightarrow T(A)$ such that $\hat{T}_{u_{0}} \hat{0}=\hat{0}+u_{0}$ and such that $\hat{T}_{u_{0}}\left(\Psi_{H}^{*}(a)\right)=$ $\Psi_{H}^{*}(a)$. Then one has trivially $\hat{T}_{u_{0}}=i d$ over the subalgebra of $T_{\omega}(\bar{A})$ isomorphic to $T_{\omega_{1}}\left(A^{1}\right)$ and we can describe the relations for equations driven by $\bar{X}$. The proof follows easily from Corollary 2.29, Theorem 3.24 and Proposition 3.19,
Corollary 3.25. Let $\mathbf{X}^{1}$ be a good sqgrm and $u_{0} \in \hat{\mathcal{L}}(A)$. Then the translation $t \mapsto$ $\hat{T}_{u_{0}} \overline{\mathbf{X}}_{0, t}$ can be described as the sum $\mathbf{U}_{0} \boxplus \overline{\mathbf{X}}$, where $\mathbf{U}_{0}$ is given by

$$
\left(\mathbf{U}_{0}\right)_{s, t}=\exp _{\otimes}\left(u_{0}(t-s)\right) .
$$

Moreover, for any given family ( $f_{\hat{0}}, f_{a}: a \in A^{1}$ ) we define $\bar{f}: T(A) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$ and $f: T\left(A_{1}\right) \rightarrow \operatorname{Vect}^{\infty}\left(\mathbb{R}^{e}\right)$ like in (19) starting from $\left(f_{\hat{0}}, f_{a}: a \in A^{1}\right)$ and ( $f_{a}: a \in A^{1}$ ) respectively and using the notiations $\hat{u_{0}}=\Phi_{H}^{*} u_{0}, \hat{\mathbf{X}}^{1}=\Phi_{H}^{*} \mathbf{X}^{1}$ equation

$$
d Y=\hat{\vec{f}}(Y) d\left(\hat{T}_{u_{0}}(\overline{\mathbf{X}})\right)
$$

is equivalent to

$$
d Y=\left(f_{\hat{0}}+\hat{\bar{f}}_{u_{0}}\right)(Y) d t+\hat{f}(Y) d \mathbf{X}^{1}, \quad d Y=\left(f_{\hat{0}}+\bar{f}_{\hat{u}_{0}}\right)(Y) d t+f(Y) d \hat{\mathbf{X}}^{1}
$$

Remark 3.26. Note that we recover in this smooth quasi-geometric rough path setting a property like the final statement of [BCFP19, Thm. 30 (ii)]: The smooth rough path increment/model $T_{v_{0}} \mathbf{Z}_{t}$ does not depend on the precise choice of the Hopf algebra homomorphism $T_{u_{0}}$, as long as $T_{u_{0}} \hat{0}=\hat{0}+u_{0}$ and $T_{u_{0}}=\mathrm{id}$ on $T_{\omega_{1}}\left(A^{1}\right)$.

## 4 Recast in an abstract Hopf algebra framework

Many of the above constructions can be formulated in the abstract framework of Hopf algebras following the approach adopted in [TZ20] and [CEMM20]. In this section, we introduce the notion smooth rough paths over a Hopf algebra and we will discuss their renormalisation. An interesting operation arises at this level: the canonical sum of rough paths, which is applied to present an alternative construction to renormalisation of branched rough paths, along the lines of [BCFP19].

### 4.1 Smooth rough paths on a Hopf algebra

In what follows we fix $(\mathcal{H}, \cdot, \Delta)$ a connected, $\mathbb{N}$-graded and locally finite commutative Hopf algebra over $\mathbb{R}$. That is to say there is a sequence of finite-dimensional vector spaces $\left\{\mathcal{H}_{n}\right\}_{n \geq 0}$ such that

$$
\mathcal{H}=\bigoplus_{n=0}^{\infty} \mathcal{H}_{n}, \quad \mathcal{H}_{0} \approx \mathbb{R}=\langle\mathbf{1}\rangle,
$$

where 1 is the unit of $\mathcal{H}$. For any given Hopf algebra in this class, we introduce two different notions of dual space: full dual $\mathcal{H}^{\prime}$ and the the graded dual $\mathcal{H}^{*}$ respectively defined by

$$
\mathcal{H}^{\prime}:=\operatorname{Hom}(\mathcal{H}, \mathbb{R})=\operatorname{Hom}\left(\bigoplus_{n=0}^{\infty} \mathcal{H}_{n}, \mathbb{R}\right), \quad \mathcal{H}^{*}:=\bigoplus_{n=0}^{\infty} \mathcal{H}_{n}^{\prime}=\bigoplus_{n=0}^{\infty} \operatorname{Hom}\left(\mathcal{H}_{n}, \mathbb{R}\right)
$$

where $V^{\prime}=\operatorname{Hom}(V, \mathbb{R})$ stands for the space of continuous $\mathbb{R}$-valued linear forms on a topological vector space $V$. There is a canonical pairing $\langle\cdot, \cdot\rangle$ between $\mathcal{H}^{\prime}$ and $\mathcal{H}$ and by equipping $\mathcal{H}^{\prime}$ is with the weak topology i.e., the weakest topology which the evaluation maps $v^{*} \mapsto\left\langle v^{*}, u\right\rangle, u \in \mathcal{H}$ the space $\mathcal{H}^{*}$ lies dense in $\mathcal{H}^{\prime}$, see [BDS16, Lem. 1.7], BDS18].

Using the grading of $\mathcal{H}$, we can also define for any $N \in \mathbb{N}$ the truncated spaces

$$
\left(\mathcal{H}^{*}\right)^{N}:=\bigoplus_{n=0}^{N} \mathcal{H}_{n}^{*}, \quad \mathcal{H}^{N}:=\bigoplus_{n=0}^{N} \mathcal{H}_{n}
$$

which yield two natural filtrations and $\left\{\left(\mathcal{H}^{*}\right)^{N}: N \in \mathbb{N}\right\},\left\{\mathcal{H}^{N}: N \in \mathbb{N}\right\}$ for the vector spaces $\mathcal{H}^{*}$ and $\mathcal{H}$. We note that $\left(\mathcal{H}^{*}\right)^{N}=\left(\mathcal{H}^{N}\right)^{*}$ and will henceforth write $\mathcal{H}^{* N}$.

The coproduct $\Delta$ on $\mathcal{H}$ induces by duality a product $\star$ in $\mathcal{H}^{\prime}$ and a fortiori on $\mathcal{H}^{*}$ defined on any $\alpha, \beta \in \mathcal{H}^{\prime}$ and $h \in \mathcal{H}$ via the identity

$$
\langle\alpha \star \beta, h\rangle:=\langle(\alpha \otimes \beta), \Delta h\rangle_{2},
$$

where $\langle\cdot, \cdot\rangle_{2}$ is the canonical pairing between $\mathcal{H}^{\prime} \otimes \mathcal{H}^{\prime}$ and $\mathcal{H} \otimes \mathcal{H}$ induced by the canonical dual pairing $\langle\cdot, \cdot\rangle$. Moreover it is also possible to define a coproduct $\Delta^{*}$ only on $\mathcal{H}^{*}$ from the identity

$$
\left\langle\Delta^{*} w, u \otimes v\right\rangle_{2}:=\langle w, u v\rangle
$$

obtaining the dual Hopf algebra ( $\mathcal{H}^{*}, \star, \Delta^{*}$ ). The coproduct $\Delta$ is compatible with the filtration and hence so is the product $\star$, which maps $\mathcal{H}^{* M} \times \mathcal{H}^{* N}$ to $\mathcal{H}^{* M+N}$. The projections $\pi_{N}: \mathcal{H} \longrightarrow \mathcal{H} / \oplus_{n=N+1}^{\infty} \mathcal{H}_{n}$ onto the quotient by the ideal $\oplus_{n=N+1}^{\infty} \mathcal{H}_{n}$ is an algebra morphism so that $\star_{N}:=\pi_{N} \star$ restricted to $\mathcal{H}_{N}$ defines a truncated product $\star_{N}: \mathcal{H}^{* N} \times \mathcal{H}^{* N} \rightarrow \mathcal{H}^{* N}$. Much in the same way as rough paths over a Hopf algebra $\mathcal{H}$ were defined, see [TZ20, CEMM20, we now define smooth rough paths and models.

Definition 4.1. We call a level- $N$ smooth $\mathcal{H}$ rough path over (in short: $N$-s $\mathcal{H}$ rp) any non zero path $\mathbf{X}:[0, T] \rightarrow \mathcal{H}^{* N}$ satisfying the following properties:
(a" i) for all times $t \in[0, T]$ and for all $h \in \mathcal{H}^{K}, k \in \mathcal{H}^{L}$, with $K+L=N$ one has

$$
\begin{equation*}
\left\langle\mathbf{X}_{t}, v \cdot w\right\rangle=\left\langle\mathbf{X}_{t}, v\right\rangle\left\langle\mathbf{X}_{t}, w\right\rangle . \tag{54}
\end{equation*}
$$

(a" ii) For every word $h \in \mathcal{H}^{N}$, the map $t \mapsto\left\langle\mathbf{X}_{t}, h\right\rangle$ is smooth.
We call level- $N$ smooth rough model over $\mathcal{H}$ (in short: $N$-s $\mathcal{H}$ rm) any map $\mathbf{X}$ : $[0, T]^{2} \rightarrow \mathcal{H}^{* N}$ which satisfies property (54) for all $\mathbf{X}_{s, t}$ as well as the following properties:
(b' i) The following abstract Chen relation holds:

$$
\begin{equation*}
\mathbf{X}_{s u} \star_{N} \mathbf{X}_{u t}=\mathbf{X}_{s, t} \tag{55}
\end{equation*}
$$

for any $s, u, t \in[0, T]$.
(b' ii) For every $h \in \mathcal{H}^{N}$, the map $t \mapsto\left\langle\mathbf{X}_{s, t}, w\right\rangle$ is smooth, for one (equivalently: all) base point(s) $s \in[0, T]$.
By smooth rough path (model) (in short: $\mathbf{s H} \mathbf{r p}$ and $\mathbf{s \mathcal { H }} \mathbf{r m}$ ), we mean a path (map) with values in $\mathcal{H}^{\prime}$ for which (54) holds for any $w, v \in \mathcal{H}$ and relation (55) holds with $\star$.

Algebraic properties of smooth rough paths on $\mathcal{H}$ are encoded in the specific Lie group (Lie algebra) structures of $\mathcal{H}^{\prime}$. Following [TZ20, Defn. 2.6], property (54) (in the case when $v, w \in \mathcal{H}$ ), amounts to $\mathbf{X}_{t}$ belonging to the group of $N$-truncated characters $G^{N}(\mathcal{H})$ or the group of characters $G(\mathcal{H})$, for any $t$. The $(G(\mathcal{H}), \star)$ is a topological Lie group when equipped with the topology of pointwise convergence, see BDS16 and by quotienting $\left(G^{N}(\mathcal{H}), \star_{N}\right)$ is a finite-dimensional Lie group. Their Lie algebras can be explicitly described by the set $\mathfrak{g}^{N}(\mathcal{H})$ of level- $N$ truncated infinitesimal characters, resp. the set $\mathfrak{g}(\mathcal{H})$ of infinitesimal characters i.e, of elements $\alpha \in \mathcal{H}^{\star N}$, resp. $\alpha \in \mathcal{H}^{\star}$ such that for all $(h, k) \in \mathcal{H}^{K} \times \mathcal{H}^{L}$ with $K+L=N$, resp. $(h, k) \in \mathcal{H}^{2}$, the following property holds:

$$
\begin{equation*}
\langle\alpha, h \cdot k\rangle=\langle\alpha, h\rangle\left\langle\mathbf{1}^{*}, k\right\rangle+\left\langle\mathbf{1}^{*}, h\right\rangle\langle\alpha, k\rangle, \tag{56}
\end{equation*}
$$

where $\mathbf{1}^{*}$ is the counit of $\mathcal{H}$. The Lie brackets $[\cdot, \cdot]_{\star_{N}}$ on $\mathfrak{g}^{N}(\mathcal{H})$ and $[\cdot, \cdot]_{\star}$ on $\mathfrak{g}(\mathcal{H})$ are given by the commutators of $\star_{N}$ and $\star$. The pair $\left(\mathfrak{g}(\mathcal{H}),[\cdot, \cdot]_{\star}\right)$ defines a topological Lie algebra, see BDS18]. A special role is played by the set of primitive elements $P\left(\mathcal{H}^{*}\right)=\mathfrak{g}(\mathcal{H}) \cap \mathcal{H}^{*}$ which is a Lie algebra with the induced Lie brackets. To relate these Lie algebras with the the character groups we introduce the exponential and truncated exponential map $\exp _{\star}: \mathcal{H}^{\prime} \rightarrow \mathcal{H}^{\prime}, \exp _{\star_{N}}: \mathcal{H}^{* N} \rightarrow \mathcal{H}^{* N}$ defined by

$$
\begin{equation*}
\exp _{\star} \mathbf{x}:=\sum_{n=0}^{\infty} \frac{\mathbf{x}^{\star n}}{n!}, \quad \exp _{\star_{N}} \mathbf{x}=\left.\sum_{n \geq 0}^{N} \frac{\mathbf{x}^{\star n}}{n!}\right|_{\mathcal{H}^{N}} \tag{57}
\end{equation*}
$$

When restricted to the Lie algebras, they induce bijections $\exp _{\star_{N}}: \mathfrak{g}^{N}(\mathcal{H}) \rightarrow G^{N}(\mathcal{H})$ and the $\exp _{\star}: \mathfrak{g}(\mathcal{H}) \rightarrow G(\mathcal{H})$, turning $G(\mathcal{H})$ into an analytic Lie group, see BDS16, Thm. 3.7 and App. B] and [BDS18, Thm. 3.9] for the properties of $\exp _{\star}$ and $G(\mathcal{H})$.

In practice, we can identify $N$-s $\mathcal{H} \operatorname{rp} \mathbf{X}_{t}$ and $N$-s $\mathcal{H} \mathrm{rm} \mathbf{X}_{s, t}$ have the same equivalence properties of their quasi geometric equivalent by considering the increments of with respect to $\star$, see equation (9). Similarly to Definition [2.5, we define the extension of a $N$-sHrp. The diagonal derivative

$$
\dot{\mathbf{X}}_{s, s}:=\left.\partial_{t}\right|_{t=s} \mathbf{X}_{s, t}=\mathbf{X}_{s}^{-1} \star \dot{\mathbf{X}}_{s}
$$

of any given $\mathbf{X}$ in sH rp lies in the Lie algebra $g(\mathcal{H})$. These properties allow to extend Theorem 2.8 on any smooth rough path over $\mathcal{H}$.
Theorem 4.2 (Fundamental Theorem of $s \mathcal{H r p}$ ). Let $N \in \mathbb{N}$. Any $\mathbf{Y}$ in $N$-sHrp uniquely extends to some $\mathbf{X}$ in sHep which is minimal in the sense that

$$
\mathbf{X}_{s}^{-1} \star \dot{\mathbf{X}}_{s} \in \mathfrak{g}^{N}(\mathcal{H})
$$

for all $s \in[0, T]$. We call $\mathcal{H} \operatorname{MinExt}(\mathbf{Y}):=\mathbf{X}$ the $\mathcal{H}$-minimal extension of $\mathbf{Y}$ and also $\mathcal{H} \operatorname{MinExt}^{\mathrm{N}^{\prime}}(\mathrm{Y}):=\pi_{N^{\prime}} \mathbf{X}$, for $N^{\prime}>N$, the $N^{\prime}$-minimal extension of $\mathbf{Y}$. Moreover, for any $[s, t] \subset[0, T]$ the associated sgrm of $\mathbf{X}_{s, t}$ only depends on $\left.\mathbf{Y}\right|_{[s, t]}$. We call this object the $\mathcal{H}$ signature of $\mathbf{Y}$ on $[s, t]$, in symbols $\operatorname{Sig}_{\mathcal{H}}\left(\left.\mathbf{Y}\right|_{[s, t]}\right)$. A sHrm $\mathbf{X}$ is called a good sH $\mathbf{H m}$ if it satisfies $\mathbf{X}=\mathcal{H} \operatorname{MinExt}(\mathbf{Y})$ for some $N$-s $\mathcal{H} r m \mathbf{Y}$.

Proof. The proof of the result goes as that of Theorem [2.8, modulo the replacement of $\otimes$ by $\star$. The only properties to check in this generalised context is the existence and uniqueness of a smooth solution for the initial value problem

$$
\begin{equation*}
\dot{\gamma}(t)=\gamma(t) \star \eta(t), \quad \gamma(0)=\mathbf{1}^{*} \tag{58}
\end{equation*}
$$

for any given smooth curve $\eta:[0, T] \rightarrow \mathfrak{g}(\mathcal{H})$. Moreover, we need to prove that for any given couple $X, \bar{X} \in G^{N+1}(\mathcal{H})$ such that $\langle X, h\rangle=\langle X, h\rangle$ for any $h \in \mathcal{H}_{N}$ then $X-\bar{X}$ belongs to the center of $G^{N+1}(\mathcal{H})$. The first property follows from the regularity in the sense of Milnor, see Mil84, of the Lie group $G(\mathcal{H})$, see [BS18, Thm. B] and references therein. The second property follows from [TZ20, Prop. 2.10].
Example 4.3. Setting $\mathcal{H}:=\left(T\left(\mathbb{R}^{d}\right), \amalg, \Delta\right)$ and $\mathcal{H}:=\left(T_{\omega}(A), \widehat{\uplus}, \Delta\right)$ we recover respectively Theorems 2.8 and 3.9. In this identification, the operation $\star$, depending on $\Delta$ is identified with the concatenation product $\otimes$ see [Reu93]. As recalled in Theorem 3.4, the Hoffman's exponential and logarithm (32) yield an isomorphism of graded Hopf algebras $\left(T\left(\mathbb{R}^{d}\right), \amalg, \Delta\right) \simeq\left(T\left(\mathbb{R}^{d}\right), \widehat{山}, \Delta\right)$. Hence the basic properties of these maps follow from the general fact that for any given Hopf algebra isomorphism $\Gamma: \mathcal{H} \rightarrow \mathcal{K}$ among two Hopf algebras $\mathcal{H}$ and $\mathcal{K}$ with the properties listed at the beginning, the adjoint map $\Gamma^{*}: \mathcal{H}^{*} \rightarrow \mathcal{K}^{*}$ is also an isomorphism. We observe also that the scalar product $\langle\cdot, \cdot \cdot\rangle$ defined on on $T\left(\mathbb{R}^{d}\right)$ and $T_{\omega}(A)$ can be used to identify the graded dual $\mathcal{H}^{*}$ with $\mathcal{H}$ by means of the Riesz lemma, so that it is not necessary to introduce the notion of graded dual in that context.

Example 4.4. Another relevant example in the context of renormalisation of rough paths arising in [BCFP19] is the Butcher-Connes-Kreimer Hopf algebra $\mathcal{H}_{B C K}\left(\mathbb{R}^{d}\right)$ consists of polynomials of rooted forests $\tau$ with nodes decorated by the finite set $\{1, \cdots, d\}$ together with the empty forest $\mathbf{1}$. A forest $f$ is graded accordingly to $|f|$, the number of its nodes and the coproduct $\Delta$ is defined on each tree $\tau$ as

$$
\begin{equation*}
\Delta(\tau)=\sum_{c} P_{c}(\tau) \otimes R_{c}(\tau), \tag{59}
\end{equation*}
$$

where the sum is taken over a specific set of admissible cuts over the tree. The result of each cut produces a polynomial of trees $P_{c}(t)$ is and $R_{c}(t)$ a tree corresponding to the root. We call the $N$-s $\mathcal{H r p}$ or $\mathrm{s} \mathcal{H}$ rp in this context level- $N$ smooth branched rough paths and smooth branched rough paths (in shorts $N$-sbrp and sbrp), see [Gub10]. The operation $\star$ induced by the coproduct (59) coincide with the so called Grossman-Larson algebra forests [GL89] and $P\left(\mathcal{H}_{B C K}^{*}\left(\mathbb{R}^{d}\right)\right)$ coincides with the free vector space $\left\langle\mathfrak{T}_{d}\right\rangle$ generated by $\mathfrak{T}_{d}$, the set of dual trees decorated by the finite set $\{1, \cdots, d\}$ and $\mathfrak{g}\left(\mathcal{H}_{B C K}\left(\mathbb{R}^{d}\right)\right)$ is the vector space $\left\langle\mathfrak{T}_{d}^{\prime}\right\rangle$ of tree series.

In adequacy with the previously results, level- $N$ smooth rough models over $\mathcal{H}$ are a special case of $\gamma$-Hölder rough paths over a Hopf algebra introduced in CEMM20.
Proposition 4.5. Every $N$-s $\mathcal{H} r m \mathbf{X}$ is a $1 / N$-regular $N$-truncated $\mathcal{H}$ rough path, see [CEMM20, Defn. 4.3] and its minimal extension $\tilde{\mathbf{X}}$ coincides with the lift of $\mathbf{X}$, as constructed in CEMM20, Thm. 4.4].

Remark 4.6. The same considerations of Remark 3.13 apply also to the class of $\gamma$-regular $N$-truncated $\mathcal{H}$ rough paths.

### 4.2 Translation of smooth rough paths on a Hopf algebra

We now extend to the framework of smooth $\mathcal{H}$-rough paths the notion of translation discussed in Theorems 2.24 and 3.21. To specify in which direction we can perform a translation, we choose a generic finite-dimensional subspace $\mathcal{D} \subset P\left(\mathcal{H}^{*}\right)$. Fixing a basis $\left\{h_{1}, \cdots, h_{e}\right\}$ of $\mathcal{D}$, we obtain a fixed set of directions to apply a translation. By assigning an element of $P\left(\mathcal{H}^{*}\right)$ to each direction, we introduce an abstract notion of translation.

Definition 4.7. Given a family of primitive elements $v=\left\{v_{i}: i=1, \cdots, e\right\} \subset \mathfrak{g}(\mathcal{H})$ we call a translation map over $\mathcal{D}$ any continuous Lie algebra homomorphism $M_{v}: \mathfrak{g}(\mathcal{H}) \rightarrow$ $\mathfrak{g}(\mathcal{H})$ such that $M_{v} h_{i}=h_{i}+v_{i}$ for any $i=1, \cdots, e$.

Remark 4.8. Using the topological properties of $\mathfrak{g}(\mathcal{H})$, if $v=\left\{v_{i}: i=1, \cdots, e\right\} \subset P\left(\mathcal{H}^{*}\right)$ to define a continuous Lie algebra homomorphism $M_{v}$ it is sufficient to have a Lie homomorphism $M_{v}: P\left(\mathcal{H}^{*}\right) \rightarrow \mathfrak{g}(\mathcal{H})$ since the Lie algebra $P\left(\mathcal{H}^{\star}\right)$ is dense in $\mathfrak{g}(\mathcal{H})$, see BDS18, Rrk. 3.11]. Note that Definition 4.7 only gives a pointwise definition for a fixed $v$. As in the cases studied in this paper, in practice, we need a map $v \rightarrow M_{v}$ with certain consistency properties like $M_{v} M_{u}=M_{v+M_{v} u}$, a condition we do not impose here. More generally we hope to investigate in future work, what properties one should impose on $\mathcal{D}$ and $v$ in a general Hopf algebra framework. A first suggestion of such a set of axioms was very recently given in Rah21, Defn.6, Defn.7].
Remark 4.9. The previous translation maps $T_{v}$ and $\hat{T}_{\hat{v}}$ in the geometric and quasigeometric setting are specific examples of translation over two different subspaces of primitive elements, i.e. the vector space $\mathbb{R}^{d}$ and the free vector space generated by $\mathfrak{A}=\left\{\Psi_{H}^{*}(a): a \in A\right\}$. This is a very specific situation one uses the specific structure of $P\left(\mathcal{H}^{*}\right)$ as a free Lie algebra, which ensures both the existence and the uniqueness of a translation map. However, it was shown in [BCFP19, Ex.9] that one can construct two different translation maps over the the same vector space $\mathcal{D}$ when $\mathcal{H}=\mathcal{H}_{B C K}\left(\mathbb{R}^{d}\right)$. The Lie structure of $\mathfrak{g}(\mathcal{H})$ is therefore not sufficient to determine a unique translation. Hence the idea of a definition which does not involve a uniqueness of the translation in its formulation.

Once given $v=\left\{v_{i}: i=1, \cdots, e\right\} \subset \mathfrak{g}(\mathcal{H})$ and a translation map $M_{v}$ over $\mathcal{D}$, we can actually uniquely extend $M_{v}$ to a continuous $\star$ morphism $M_{v}: \mathcal{H}^{\prime} \rightarrow \mathcal{H}^{\prime}$ the full translation map which we denote in the same way. The extension is purely algebraic and follows by standard Milnor-Moore theorem MM65. Indeed any translation map defines uniquely a Lie algebra morphism $M_{v}: P\left(\mathcal{H}^{*}\right) \rightarrow \mathcal{H}^{\prime}$ which is compatible with the product $\star$. Using the universal property of the universal enveloping algebra $\mathcal{U}\left(P\left(\mathcal{H}^{*}\right)\right)$ and the Milnor-Moore theorem, the map $M_{v}$ uniquely extends to a $\star$ morphism $M_{v}: \mathcal{H}^{*} \rightarrow \mathcal{H}^{\prime}$ and by density it can be defined over all $\mathcal{H}^{\prime}$. The map $M_{v}$ to perform translation of $\mathrm{s} \mathcal{H} \mathrm{rms}$ Theorem 2.24 and Proposition 3.23 ,

Theorem 4.10. Given a sH$r p(s \mathcal{H} r m)$ a family $v=\left\{v_{i}: i=1, \cdots, e\right\} \subset \mathfrak{g}(\mathcal{H})$ and a translation map $M_{v}$ over $\mathcal{D}$ the composition $M_{v}\left(\mathbf{X}_{t}\right),\left(M_{v}\left(\mathbf{X}_{s, t}\right)\right)$ is again a sHrp (sHrm) which coincides (up to increments) with the solution of

$$
\begin{equation*}
\dot{\mathbf{Z}}_{t}=\mathbf{Z}_{t} \star\left(M_{v} \dot{\mathbf{X}}_{t, t}\right), \quad \mathbf{Z}_{0}=M_{v} \mathbf{X}_{0} . \tag{60}
\end{equation*}
$$

The same result applies to good sH rms when $v \subset P\left(\mathcal{H}^{*}\right)$. Supposing also that for any $N \in \mathbb{N}$ there exists an integer $L \geq N$ depending on $v$ such that $M_{v}: \mathcal{H}^{* N} \rightarrow \mathcal{H}^{* L}$, then for any $N$-sH $r p Y$ the unique solution to

$$
\dot{\mathbf{W}}_{t}=\mathbf{W}_{t} \star_{L}\left(M_{v} \dot{\mathbf{Y}}_{t, t}\right), \quad \mathbf{W}_{0}=\mathbf{1}^{*}
$$

defines a L-sHrm, given by $\mathbf{W}_{s, t}=\mathbf{W}_{s}^{-1} \otimes_{M} \mathbf{W}_{t}$, which we call $\mathcal{M}_{v}[\mathbf{Y}]$. Moreover, we have the explicit form

$$
\begin{equation*}
\mathcal{M}_{v}[\mathbf{Y}]_{s, t}=M_{v}^{L}\left(\mathbf{Y}_{s, t}^{L}\right)=\pi_{L} M_{v}\left(\mathbf{X}_{s, t}\right) \tag{61}
\end{equation*}
$$

with algebra endomorphism $M_{v}^{L}:=\pi_{M} M_{v} i^{M}$ of $\left(\mathcal{H}^{*} M, \star_{M}\right)$, using the (linear) embedding $\mathfrak{i}^{M}: \mathcal{H}^{*} M \rightarrow \mathcal{H}^{\prime}$, and $\mathbf{Y}^{M}=\mathcal{H} \operatorname{MinExt}^{M}(\mathbf{Y})$.

Proof. The theorem is then concluded by checking that $M_{v} \mathbf{X}$ and $\mathcal{M}_{v}[\mathbf{Y}]$ solves the equation (60) and (61) for any $\mathbf{s H} \operatorname{rp} \mathbf{X}$ and $N$-s $\mathcal{H} r p Y$. This last check follows like in the proof of Theorem 2.24 by replacing $\otimes$ with $\star$.

As direct application of Theorem 4.10, we present a characterisation of the renormalisation of branched rough paths introduced in [BCFP19] when $\mathcal{H}=\mathcal{H}_{B C K}\left(\mathbb{R}^{d}\right)$. In that case, the authors provided the existence a translation map $M_{v}: \mathfrak{T}_{d} \rightarrow \mathfrak{T}_{d}$ over the subspace $\mathcal{D}$ generated by the dual trees $\left\{\bullet_{i}^{*}: i=1, \cdots, d\right\}$ by using a specific property of the Lie algebra $\left(\left\langle\mathfrak{T}_{d}\right\rangle,[,]_{\star}\right)$, which we briefly sketch.

Recall that a (left) pre-Lie algebra operation is a vector space $V$ equipped with a bilinear map $\curvearrowright: V \otimes V \rightarrow V$ whose associator

$$
(x, y, z)=(x \curvearrowright y) \curvearrowright z-x \curvearrowright(y \curvearrowright z)
$$

is invariant under the exchange of the two variables $y$ and $z$, see [Man11]. Given a pre-Lie algebra, one can construct a Lie bracket via its anti-symmetrisation

$$
[x, y]_{\curvearrowright}=(x \curvearrowright y)-(y \curvearrowright x) .
$$

It turns out that $\left\langle\mathfrak{T}_{d}\right\rangle$ admits an explicit pre-Lie algebra structure $\curvearrowright$ which as well as satisfying $[,]_{\curvearrowright}=[,]_{\star}$, also has the property that $\left(\left\langle\mathfrak{T}_{d}\right\rangle, \curvearrowright\right)$ is isomorphic to the free Pre-Lie algebra over $d$ elements, see [CL01].

Thanks to this property for any given family of tree series $v=\left(v_{1}, \cdots, v_{d}\right) \subset \mathfrak{T}_{d}^{\prime}$, it is then possible to fix a unique translation map $M_{v}:\left\langle\mathfrak{T}^{\prime}{ }_{d}\right\rangle \rightarrow\left\langle\mathfrak{T}^{\prime}{ }_{d}\right\rangle$ which satisfies

$$
\begin{equation*}
M_{v} \bullet_{i}^{*}=\bullet_{i}^{*}+v_{i}, \quad M_{v}(x \curvearrowright y)=\left(M_{v} x\right) \curvearrowright\left(M_{v} y\right), \tag{62}
\end{equation*}
$$

for any $i=1, \cdots, d$. It follows from the standard properties of the operation $\curvearrowright$ and the grossmann-Larson product on the grading on trees $|\cdot|$ that for any given $v=\left(v_{1}, \cdots, v_{d}\right) \subset \mathfrak{T}_{d} M_{v}$ maps $\mathcal{H}_{B C K}^{* N}\left(\mathbb{R}^{d}\right)$ to $\mathcal{H}_{B C K}^{* L}\left(\mathbb{R}^{d}\right)$ where $L=N \cdot N^{\prime}$ with $N^{\prime}$ the smallest integer such that $\left|v_{i}\right| \leq N^{\prime}$ for any $i=1, \cdots, d$. From Theorem 4.10 we deduce the following corollary.

Corollary 4.11. Given a sbrp (sbrm) $\mathbf{X}$ and $v=\left(v_{1}, \cdots v_{d}\right) \subset \mathfrak{T}_{d}^{\prime}$, the composition $M_{v}\left(\mathbf{X}_{t}\right)\left(M_{v}\left(\mathbf{X}_{s, t}\right)\right)$ with $M_{v}$ uniquely defined by (62) is again a sqgrp (sqgrm) which coincides (up to increments) with the solution of

$$
\begin{equation*}
\dot{\mathbf{Z}}_{t}=\mathbf{Z}_{t} \otimes\left(M_{v} \dot{\mathbf{X}}_{t, t}\right), \quad \mathbf{Z}_{0}=M_{v} \mathbf{X}_{0} \tag{63}
\end{equation*}
$$

The same result applies to good sqgrms when $v \subset \mathfrak{T}_{d}$. Under the same restriction on $v$, for any $N$-sbrm $\mathbf{Y}$ the path $t \rightarrow \mathcal{M}_{u}[\mathbf{Y}]_{0, t}$ is a L-sbrp coinciding with the solution of

$$
\begin{equation*}
\dot{\mathbf{W}}_{t}=\mathbf{W}_{t} \otimes_{L}\left(M_{v} \dot{\mathbf{Y}}_{t, t}\right), \quad \mathbf{W}_{0}=\mathbf{1}^{*}, \tag{64}
\end{equation*}
$$

where $L \in \mathbb{N}$ is the smallest integer such that $\left|v_{i}\right| \leq N^{\prime}$ for any $i=1, \cdots, d$.
Remark 4.12. Conditions (62) identify uniquely a full translation map $M_{v}$ whose explicit calculation on forest is not direct. In BCFP19] the authors obtained a dual description of the dual map $M_{v}^{*}: \mathcal{H}_{B C K}\left(\mathbb{R}^{d}\right) \rightarrow \mathcal{H}_{B C K}\left(\mathbb{R}^{d}\right)$ using coalgebraic tools related to extraction and contraction of trees but an explicit expression of $M_{v}$ is still unknown. Looking at equations (63), (64) we realize that in case of smooth branched rough paths it is sufficient to compute $M_{v}$ only on trees, which should slightly simplify the computations. Moreover, when $M_{v}$ coincides with the addition in (69), then we obtain an explicit equation which does not involve any coalgebraic tool.
Remark 4.13. Concerning the question of defining translation maps for more general Hopf algebras $\mathcal{H}$, the example of the free pre-Lie algebra points towards the general approach of using some 'free' algebraic structure on the Lie algebra $P\left(\mathcal{H}^{*}\right)$, if it is available, to define the translation map as a universal homomorphism of the free object. Ideas to this regard were proposed in [Pre21, Defn.5.2.1] and Rah21, Defn.8]. In the latter, this basic idea is also applied to the case of the free post-Lie algebra taking the role of $P\left(\mathcal{H}^{*}\right)$.

### 4.3 Canonical sum and minimal coupling of smooth rough paths

We pass now to the notion of sum in the generalised context of $\mathrm{s} \mathcal{H} \mathrm{rms}$, extending Definition 2.15 to a generic Hopf algebra as before.

Definition 4.14. For any fixed sH $\mathcal{H} \mathbf{~ X}, \mathbf{Y}:[0, T]^{2} \rightarrow \mathcal{H}^{\prime}$ let $t \mapsto \mathbf{Z}_{t} \in \mathcal{H}^{\prime}$ be the Cartan development of $\dot{\mathbf{X}}_{s, s}+\dot{\mathbf{Y}}_{s, s}$, i.e. the unique solution to

$$
\dot{\mathbf{Z}}_{t}=\mathbf{Z}_{t} \star\left(\dot{\mathbf{X}}_{t, t}+\dot{\mathbf{Y}}_{t, t}\right), \quad \mathbf{Z}_{0}=\mathbf{1}^{*}
$$

We then write $\mathbf{Z}:=\mathbf{X} \boxplus \mathbf{Y}$ for the associated sgrm and call it the canonical sum of $\mathbf{X}$ and $\mathbf{Y}$. For any $\lambda \in \mathbb{R}$ we define also the sgrm $\mathbf{Z}=\lambda \boxtimes \mathbf{X}$ via the Cartan development of $\lambda \dot{\mathbf{X}}_{s, s}$, we call it the canonical scalar multiplication.

Remark 4.15. This scalar multiplication yields a new scaling device for smooth rough paths which strongly differs from the well-known dilation $\delta_{\lambda}$, where the latter is defined for any $x \in \mathcal{H}_{n}$ by $\left\langle\delta_{\lambda} \mathbf{X}_{s, t}, x\right\rangle=\left\langle\mathbf{X}_{s, t}, \lambda^{n} x\right\rangle=\lambda^{n}\left\langle\mathbf{X}_{s, t}, x\right\rangle$. In contrast to the pointwise scaling $\delta_{\lambda}$, this new scaling is a dynamical way to scale a smooth rough path. However, we observe that $(0 \backsim \mathbf{X})_{s, t}=\delta_{0} \mathbf{X}_{s, t}=\mathbf{1}^{*}$ and that $\overline{\mathbf{X}}$ (the backward rough path), $\delta_{-1} \mathbf{X}$ and $(-1) \boxtimes \mathbf{X}$ are pairwise distinct. We expect this to have interesting applications to the theory of signatures of rough paths in the geometric setting.

From the vector space structure of the Lie algebra $\mathfrak{g}(\mathcal{H})$, we easily deduce that the set of all smooth $\mathcal{H}$ rough models forms itself a vector space when equipped with the sum $\boxplus$

[^3]and the scalar multiplication $\downarrow$, with the set of all good smooth $\mathcal{H}$ rough models forming a subspace. In particular, for any real numbers $\lambda_{1}, \lambda_{2}, \lambda$ we have
$\mathbf{X} \boxplus \mathbf{Y}=\mathbf{Y} \boxplus \mathbf{X}, \quad \lambda \backsim(\mathbf{X} \boxplus \mathbf{Y})=(\lambda \boxtimes \mathbf{X}) \boxplus(\lambda \boxtimes \mathbf{Y}), \quad\left(\lambda_{1}+\lambda_{2}\right) \boxtimes \mathbf{X}=\left(\lambda_{1} \boxtimes \mathbf{X}\right) \boxplus\left(\lambda_{2} \boxtimes \mathbf{X}\right)$.
This is very much in contrast to the spaces of $\gamma$-Hölder or bounded $p$-variation rough paths for $\gamma \leq 1 / 2$ and $p \geq 2$, where it is basic folklore by now that such a vector space structure does not exist in any meaningful sense.

It is then possible to characterise the sum $\boxplus$ via the group operation $\star$ up to some small remainder. The following theorem is an extension in a Hopf algebra and smooth case of Lyo98, Section 3.3.1 B], which was stated in the context of geometric $p$-variation rough paths.

Proposition 4.16. For any fixed couple of $s \mathcal{H r m s} \mathbf{X}, \mathbf{Y}$ a map Z: $[0, T]^{2} \rightarrow G(\mathcal{H})$ coincides with $\mathbf{X} \boxplus \mathbf{Y}$ if and only if $\mathbf{Z}$ satisfies $\mathbf{Z}_{s, t}=\mathbf{Z}_{s, u} \star \mathbf{Z}_{u, t}$ for $s, u, t \in[0, T]$ and one has for any $s \in[0, T]$

$$
\begin{equation*}
\mathbf{Z}_{s, t}=\mathbf{X}_{s, t} \star \mathbf{Y}_{s, t}+R_{s, t}, \tag{65}
\end{equation*}
$$

for some $R_{s, t} \in \mathcal{H}^{\prime}$ such that for all $x \in \mathcal{H}$ one has $\left\langle R_{s, t}, x\right\rangle=o(|t-s|)$ as $t \rightarrow s$. Moreover, we have the relations

$$
\begin{equation*}
\mathbf{X}_{s, t} \star \mathbf{Y}_{s, t}=\mathbf{Y}_{s, t} \star \mathbf{X}_{s, t}+r_{s, t}=\mathbf{X}_{s, t}+\mathbf{Y}_{s, t}-\mathbf{1}^{*}+r_{s, t}^{\prime} \tag{66}
\end{equation*}
$$

for some $r_{s, t}, r_{s, t}^{\prime} \in \mathcal{H}^{\prime}$ such that for all $x \in \mathcal{H}$ one has $\left\langle r_{s, t}, x\right\rangle,\left\langle r_{s, t}^{\prime}, x\right\rangle=o(|t-s|)$ as $t \rightarrow s$.

Proof. Let us start by formula (65). We fix $\mathbf{Z}:[0, T]^{2} \rightarrow G(\mathcal{H})$ any map with $\mathbf{Z}_{s, t}=$ $\mathbf{Z}_{s, u} \star \mathbf{Z}_{u, t}$ and for all $x \in \mathcal{H}$ one has $\left\langle\mathbf{Z}_{s, t}-\mathbf{X}_{s, t} \star \mathbf{Y}_{s, t}, x\right\rangle=o(|t-s|)$. Then, by considering the path $t \rightarrow \mathbf{Z}_{0, t}$ and fixing $s \in[0, T]$, we use the continuity of the map $\mathcal{H}^{\prime} \ni \mathbf{x} \mapsto \mathbf{Z}_{0, s} \star \mathbf{x}$, for the weak convergence with respect to the duality pairing of $\mathcal{H}^{\prime}$ with $\mathcal{H}$ to have the equalities

$$
\begin{aligned}
\lim _{t \rightarrow s} \frac{\mathbf{Z}_{0, t}-\mathbf{Z}_{0, s}}{t-s} & =\mathbf{Z}_{0 s} \star \lim _{t \rightarrow s} \frac{\mathbf{Z}_{s, t}-\mathbf{1}^{*}}{t-s}=\mathbf{Z}_{0, s} \star \lim _{t \rightarrow s} \frac{\mathbf{X}_{s, t} \star \mathbf{Y}_{s, t}-\mathbf{1}^{*}}{t-s} \\
& =\left.\mathbf{Z}_{0, s} \star \partial_{t}\right|_{t=s}\left(\mathbf{X}_{s, t} \star \mathbf{Y}_{s, t}\right)=\mathbf{Z}_{0 s} \star\left(\dot{\mathbf{X}}_{s, s}+\dot{\mathbf{Y}}_{s, s}\right) .
\end{aligned}
$$

Thus the path $s \mapsto \mathbf{Z}_{0, s}$ is differentiable with derivative $s \mapsto \mathbf{Z}_{0, s} \star\left(\dot{\mathbf{X}}_{s, s}+\dot{\mathbf{Y}}_{s, s}\right)$, implying $\dot{\mathbf{Z}}_{s, s}=\dot{\mathbf{X}}_{s, s}+\dot{\mathbf{Y}}_{s, s}$, i.e. $\mathbf{Z}=\mathbf{X} \boxplus \mathbf{Y}$. On the other hand, assuming that $\mathbf{Z}$ is given by $\mathbf{X} \boxplus \mathbf{Y}$, for all $x \in \mathcal{H}$ there are remainders $\theta_{s, t}^{x}=o(|t-s|), \theta_{s, t}^{x}=o(|t-s|)$ such that

$$
\begin{align*}
\left\langle\mathbf{Z}_{s, t}, x\right\rangle & =\left\langle\dot{\mathbf{Z}}_{s, s}, x\right\rangle(t-s)+\theta_{s, t}^{x} \\
& =\left\langle\dot{\mathbf{X}}_{s, s}+\dot{\mathbf{Y}}_{s, s}, x\right\rangle(t-s)+\theta_{s, t}^{x}  \tag{67}\\
& =\left\langle\mathbf{X}_{s, t}+\mathbf{Y}_{s, t}-\mathbf{1}^{*}, x\right\rangle+\theta_{s, t}^{x}
\end{align*}
$$

Therefore the equivalence (65) will follow from equivalence (66). Recall that for $\mathbf{X}_{s, t}, \mathbf{Y}_{s, t}$ in $G(\mathcal{H})$ and $x$ in $\mathcal{H}$, we have $\left\langle\mathbf{X}_{s, t} \star \mathbf{Y}_{s, t}, x\right\rangle=\left\langle\mathbf{X}_{s, t} \hat{\otimes} \mathbf{Y}_{s, t}, \Delta x\right\rangle$, where $\hat{\otimes}$ is the completed external tensor product, and furthermore $\left\langle\mathbf{X}_{s, t}, \mathbf{1}\right\rangle=\left\langle\mathbf{Y}_{s, t}, \mathbf{1}\right\rangle=1$. Hence,

$$
\begin{equation*}
\left\langle\mathbf{X}_{s, t} \star \mathbf{Y}_{s, t}, \mathbf{1}\right\rangle=\left\langle\mathbf{Y}_{s, t} \star \mathbf{X}_{s, t}, \mathbf{1}\right\rangle=\left\langle\mathbf{Y}_{s, t}+\mathbf{X}_{s, t}-\mathbf{1}^{*}, \mathbf{1}\right\rangle=1 \tag{68}
\end{equation*}
$$

Furthermore, for every $x \in \mathcal{H}^{\geq 1}=\sum_{n=1}^{\infty} \mathcal{H}_{n}$, i.e. $\left\langle\mathbf{1}^{*}, x\right\rangle=0$, we write $\Delta x=\mathbf{1} \otimes x+x \otimes$ $\mathbf{1}+\sum_{i=1}^{n} y_{i} \otimes z_{i}$ for some $n$ and $y_{i}, z_{i} \in \mathcal{H} \geq 1$, thereby obtaining

$$
\begin{aligned}
\left\langle\mathbf{X}_{s, t} \star \mathbf{Y}_{s, t}, x\right\rangle & =\left\langle\mathbf{X}_{s, t} \hat{\otimes} \mathbf{Y}_{s, t}, \Delta x\right\rangle=\left\langle\mathbf{X}_{s, t}, x\right\rangle+\left\langle\mathbf{Y}_{s, t}, x\right\rangle+\sum_{i=1}^{n}\left\langle\mathbf{X}_{s, t}, y_{i}\right\rangle\left\langle\mathbf{Y}_{s, t}, z_{i}\right\rangle \\
& =\left\langle\mathbf{X}_{s, t}+\mathbf{Y}_{s, t}-\mathbf{1}^{*}, x\right\rangle+\underbrace{\sum_{i=1}^{n}\left\langle\mathbf{X}_{s, t}, y_{i}\right\rangle\left\langle\mathbf{Y}_{s, t}, z_{i}\right\rangle}_{o(|t-s|)}
\end{aligned}
$$

The fact that this last term is of order $o(|t-s|)$ follows from the smoothness of $\mathbf{X}$ and $\mathbf{Y}$ and $\left\langle\mathbf{X}_{t, t}, y_{i}\right\rangle=\left\langle\mathbf{Y}_{t, t}, z_{i}\right\rangle=0$, which for any index $i$, actually yields the existence of a constant $C_{i}>0$ such that $\left|\left\langle\mathbf{X}_{s, t}, y_{i}\right\rangle\left\langle\mathbf{Y}_{s, t}, z_{i}\right\rangle\right| \leq C_{i}(t-s)^{2}$ for all $t, s \in[0, T]$. Similarly, one shows that $\left\langle\mathbf{Y}_{s, t} \star \mathbf{X}_{s, t}-\mathbf{X}_{s, t}-\mathbf{Y}_{s, t}+\mathbf{1}^{*}, x\right\rangle=o(|t-s|)$.
Remark 4.17. As pointed out in Remark 2.17 with reference to Lyo98, Section 3.3.1 B] for smooth geometric rough paths, we strongly conjecture that one can add the minimal extension of a $N$-s $\mathcal{H}$ rm X to any general $\gamma$-Hölder $\mathcal{H}$ rough path $\mathbf{W}$ for any $\gamma \in(0,1)$, via the requirement $\left\langle(\mathbf{X} \boxplus \mathbf{W})_{s, t}, x\right\rangle=\left\langle\mathbf{X}_{s, t} \star \mathbf{W}_{s, t}, x\right\rangle+o(|t-s|)$ using the sewing lemma.

The canonical sum can now be used to define a minimal coupling of smooth rough paths in the situation where we have connected graded commutative Hopf algebras $\left(\mathfrak{H}^{i}\right)_{i=1, \ldots, m}$ of the form

$$
\mathfrak{H}^{i}=\bigoplus_{j=0}^{\infty} \mathfrak{H}_{j}^{i}
$$

together with embeddings $\iota_{i}:\left(\mathfrak{H}^{i}\right)^{*} \rightarrow(\mathcal{H})^{*}$ which are injective graded Hopf algebra homorphisms, such that the sum of the non-unital parts (the kernels of the counit of $\mathcal{H}^{*}$ ) of the images $\hat{\mathfrak{H}}^{i}:=\iota_{i}\left(\mathfrak{H}^{i}\right)^{*}$ and $\hat{\mathfrak{H}}_{j}^{i}:=\iota_{i}\left(\mathfrak{H}^{i}\right)_{j}^{*}$ inside $\mathcal{H}^{*}$ forms a direct product, i.e.

$$
\sum_{i=1}^{m} \hat{\mathfrak{H}}^{i, \geq 1}=\bigoplus_{i=1}^{m} \hat{\mathfrak{H}}^{i, \geq 1}
$$

where $\hat{\mathfrak{H}}^{i, \geq 1}:=\bigoplus_{j=1}^{\infty} \hat{\mathfrak{H}}_{j}^{i}=\left\{x \in \hat{\mathfrak{H}}^{i}:\langle x, \mathbf{1}\rangle=0\right\}$, and such that the first level of $\mathcal{H}^{*}=$ $\oplus_{n=0}^{\infty} \mathcal{H}_{n}^{*}$ is actually spanned by the first levels of the $\hat{\mathfrak{H}}^{i}$, i.e.

$$
\mathcal{H}_{1}^{*}=\bigoplus_{i=1}^{m} \hat{\mathfrak{H}}_{1}^{i}
$$

Any s.j ${ }^{i} \mathrm{rp} \mathbf{X}^{i}:[0, T] \rightarrow G\left(\mathfrak{H}^{i}\right)$ is trivially mapped via the embedding $\iota_{i}$ and extended by continuity inside $\mathcal{H}^{\prime}$ to a $\operatorname{sHrp} \iota_{i} \mathbf{X}^{i}:[0, T] \rightarrow G(\mathcal{H})$. We define then the minimal coupling of the $\mathbf{X}^{i}$ as

$$
\left(\mathbf{X}^{1}, \ldots, \mathbf{X}^{m}\right)_{\min }:=\iota_{1} \mathbf{X}^{1} \boxplus \cdots \boxplus \iota_{m} \mathbf{X}^{m} .
$$

As a special case, we consider two connected graded commutative Hopf algebras $\mathfrak{H}^{0}$, $\mathfrak{H}^{1}$ with embedded graded dual spaces $\hat{\mathfrak{H}}^{0}=\iota_{0}\left(\mathfrak{H}^{0}\right)^{*}, \hat{\mathfrak{H}}^{1}=\iota_{1}\left(\mathfrak{H}^{1}\right)^{*}$ such that

$$
\hat{\mathfrak{H}}^{0, \geq 1}+\hat{\mathfrak{H}}^{1, \geq 1}=\hat{\mathfrak{H}}^{0, \geq 1} \oplus \hat{\mathfrak{H}}^{1, \geq 1}, \quad \mathcal{H}_{1}^{*}=\hat{\mathfrak{H}}_{1}^{0} \oplus \hat{\mathfrak{H}}_{1}^{1} .
$$

By taking $\hat{\mathfrak{H}}^{0}$ the unital subalgebra generated by just a single element $x_{0} \in(\mathcal{H})_{1}^{*}$ (which is automatically a graded sub Hopf algebra of $\left.\mathcal{H}^{*}\right)$, the only smooth rough models $\mathbf{X}^{0}$ : $[0, T] \rightarrow G\left(\mathfrak{H}^{0}\right)$ are of the form

$$
\iota_{0} \dot{\mathbf{X}}_{s, s}^{0}=\psi(s) x_{0}
$$

for some smooth $\psi:[0, T] \rightarrow \mathbb{R}$. Then one easily checks that

$$
\iota_{0} \mathbf{X}_{s, t}^{0}=\exp _{\star}\left(\left(\int_{s}^{t} \psi(u) d u\right) x_{0}\right)
$$

By taking $\psi=1$, we find that for any $\mathrm{s} \mathfrak{H}^{1} \mathrm{rp} \mathbf{X}^{1}:[0, T] \rightarrow G\left(\mathfrak{H}^{1}\right)$ the minimal coupling $\overline{\mathbf{X}}:=\left(\mathbf{X}^{0}, \mathbf{X}^{1}\right)_{\text {min }}$ is uniquely determined by

$$
\dot{\overline{\mathbf{X}}}_{s, s}=x_{0}+\dot{\mathbf{X}}_{s, s}^{1} .
$$

In this situation, we can directly reinterpret $\overline{\mathbf{X}}$ as a specific type of full translation.
Corollary 4.18. Let $\mathbf{X}^{1}$ be a good $s \mathcal{H} r m, v_{0} \in \mathfrak{g}(\mathcal{H})$ and $M_{v_{0}}: \mathcal{H}^{*} \rightarrow \mathcal{H}^{*}$ a full translation map such that $M_{v_{0}} x_{0}=x_{0}+v_{0}$ and $M_{v_{0}}$ restricted to $\iota_{1}\left(\mathfrak{H}^{1}\right)^{*}$ is the identity. Then the translation $t \mapsto M_{v_{0}} \overline{\mathbf{X}}_{0, t}$ can be described as the sum $\mathbf{V}_{0} \boxplus \overline{\mathbf{X}}$, where $\mathbf{V}_{0}$ is given by

$$
\left(\mathbf{V}_{0}\right)_{s, t}=\exp _{\star}\left(v_{0}(t-s)\right) .
$$

Proof. We obviously have

$$
\begin{equation*}
\left.\partial_{t}\right|_{t=s}\left(M_{v_{0}} \overline{\mathbf{X}}_{s, t}\right)=M_{v_{0}} \dot{\overline{\mathbf{X}}}_{s, s}=x_{0}+v_{0}+\dot{\mathbf{X}}_{s, s}^{1}=v_{0}+\dot{\overline{\mathbf{X}}}_{s, s}, \tag{69}
\end{equation*}
$$

which shows $M_{v_{0}} \overline{\mathbf{X}}=\mathbf{V}_{0} \boxplus \overline{\mathbf{X}}$.
Remark 4.19. In this general connected graded commutative Hopf algebra setting, we once again recover a property similar to the final statement of [BCFP19, Thm. 30 (ii)]: the $\mathrm{s} \mathcal{H r p} M_{v_{0}} \overline{\mathbf{X}}_{t}$ does not depend on the precise choice of the Hopf algebra homomorphism $M_{v_{0}}$, as long as $M_{v_{0}} x_{0}=x_{0}+v_{0},\left.M_{v_{0}}\right|_{\hat{\mathfrak{H}}^{1}}=\mathrm{id}$.

## References

[AFS19] Carlos Améndola, Peter K. Friz, and Bernd Sturmfels. Varieties of signature tensors. Forum Math. Sigma, 7:e10, 2019.
[BCE20] Yvain Bruned, Charles Curry, and Kurusch Ebrahimi-Fard. Quasi-shuffle algebras and renormalisation of rough differential equations. Bull. Lond. Math. Soc., 52(1):43-63, 2020.
[BCFP19] Yvain Bruned, Ilya Chevyrev, Peter K. Friz, and Rosa Preiß. A rough path perspective on renormalization. J. Funct. Anal., 277(11):108283, 60, 2019.
[BDS16] Geir Bogfjellmo, Rafael Dahmen, and Alexander Schmeding. Character groups of Hopf algebras as infinite-dimensional Lie groups. Ann. Inst. Fourier (Grenoble), 66(5):2101-2155, 2016.
[BDS18] Geir Bogfjellmo, Rafael Dahmen, and Alexander Schmeding. Overview of (pro-)Lie group structures on Hopf algebra character groups. In Discrete mechanics, geometric integration and Lie-Butcher series, volume 267 of Springer Proc. Math. Stat., pages 287-314. Springer, Cham, 2018.
[Bel20] Carlo Bellingeri. Quasi-geometric rough paths and rough change of variable formula. arXiv preprint arXiv:2009.00903, page 46 pp., 2020.
[BHZ19] Yvain Bruned, Martin Hairer, and Lorenzo Zambotti. Algebraic renormalisation of regularity structures. Invent. Math., 215(3):1039-1156, 2019.
[Bru20] Yvain Bruned. Renormalisation from non-geometric to geometric rough paths. arXiv e-prints 2007.14385, page 19 pp., 2020.
[BS18] Geir Bogfjellmo and Alexander Schmeding. The geometry of characters of Hopf algebras. In Computation and combinatorics in dynamics, stochastics and control, volume 13 of Abel Symp., pages 159-185. Springer, Cham, 2018.
[Car72] Pierre Cartier. On the structure of free Baxter algebras. Advances in Math., 9:253-265, 1972.
[CDL15] Thomas Cass, Bruce K. Driver, and Christian Litterer. Constrained rough paths. Proc. Lond. Math. Soc. (3), 111(6):1471-1518, 2015.
[CEMM20] Charles Curry, Kurusch Ebrahimi-Fard, Dominique Manchon, and Hans Z. Munthe-Kaas. Planarly branched rough paths and rough differential equations on homogeneous spaces. J. Differential Equations, 269(11):9740-9782, 2020.
[CEMW14] Charles Curry, Kurusch Ebrahimi-Fard, Simon J.A. Malham, and Anke Wiese. Lévy processes and quasi-shuffle algebras. Stochastics, 86(4):632642, 2014.
[CGMe20] Laura Colmenarejo, Francesco Galuppi, and Mateusz Michał ek. Toric geometry of path signature varieties. Adv. in Appl. Math., 121:102102, 35, 2020.
[CGPZ20] Pierre Clavier, Li Guo, Sylvie Paycha, and Bin Zhang. Renormalisation and locality: branched zeta values. In Algebraic Combinatorics, Resurgence, Moulds and Applications (CARMA), pages 85-132. EMS Publishing House, 2020.
[Che54] Kuo-Tsai Chen. Iterated integrals and exponential homomorphisms. Proc. London Math. Soc. (3), 4:502-512, 1954.
[CK16] Ilya Chevyrev and Andrey Kormilitzin. A primer on the signature method in machine learning. arXiv preprint arXiv:1603.03788, page $46 \mathrm{pp} ., 2016$.
[CL01] Frédéric Chapoton and Muriel Livernet. Pre-Lie algebras and the rooted trees operad. Internat. Math. Res. Notices, 8:395-408, 2001.
[CLT19] Elena Celledoni, Pål Erik Lystad, and Nikolas Tapia. Signatures in shape analysis: an efficient approach to motion identification. In Geometric science of information, volume 11712 of Lecture Notes in Comput. Sci., pages 21-30. Springer, Cham, 2019.
[CQRV02] Chryssomalis Chryssomalakos, Hernando Quevedo, Marcos Rosenbaum, and José D. Vergara. Normal coordinates and primitive elements in the Hopf algebra of renormalization. Comm. Math. Phys., 225(3):465-485, 2002.
[Dav08] Alexander M. Davie. Differential equations driven by rough paths: an approach via discrete approximation. Appl. Math. Res., page 40 pp., 2008. [Issue information previously given as no. 2 (2007)].
[DET20] Joscha Diehl, Kurusch Ebrahimi-Fard, and Nikolas Tapia. Time-warping invariants of multidimensional time series. Acta Appl. Math., 170:265-290, 2020.
[EMPW15] Kurusch Ebrahimi-Fard, Simon J. A. Malham, Frederic Patras, and Anke Wiese. The exponential Lie series for continuous semimartingales. Proc. A., 471(2184):20150429, 19, 2015.
[FG16] Peter K. Friz and Paul Gassiat. Geometric foundations of rough paths. In Geometry, Analysis and Dynamics on sub-Riemannian Manifolds, pages 171-210. European Mathematical Society Publishing House, 2016.
[FH20] Peter K. Friz and Martin Hairer. A Course on Rough Paths, with introduction to Regularity Structures (2nd extended edition). Springer International Publishing, 2020.
[FP20] Loïc Foissy and Frédéric Patras. Lie theory for quasi-shuffle bialgebras. In Periods in quantum field theory and arithmetic, volume 314 of Springer Proc. Math. Stat., pages 483-540. Springer, Cham, 2020.
[FS17] Peter K. Friz and Atul Shekhar. General rough integration, Lévy rough paths and a Lévy-Kintchine-type formula. Ann. Probab., 45(4):2707-2765, 2017.
[FV10] Peter K. Friz and Nicolas Victoir. Multidimensional stochastic processes as rough paths, volume 120 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2010. Theory and applications.
[Gal19] Francesco Galuppi. The rough Veronese variety. Linear Algebra Appl., 583:282-299, 2019.
[GL89] Robert Grossman and Richard G. Larson. Hopf-algebraic structure of families of trees. J. Algebra, 126(1):184-210, 1989.
[Gub10] Massimiliano Gubinelli. Ramification of rough paths. J. Differential Equations, 248(4):693-721, 2010.
[Hai13] Martin Hairer. Solving the KPZ equation. Ann. of Math. (2), 178(2):559664, 2013.
[Hai14] Martin Hairer. A theory of regularity structures. Invent. Math., 198(2):269504, 2014.
[HK15] Martin Hairer and David Kelly. Geometric versus non-geometric rough paths. Ann. Inst. Henri Poincaré Probab. Stat., 51(1):207-251, 2015.
[Hof00] Michael E. Hoffman. Quasi-shuffle products. J. Algebraic Combin., 11(1):4968, 2000.
[IN99] Arieh Iserles and Syvert P. Nørsett. On the solution of linear differential equations in Lie groups. R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci., 357(1754):983-1019, 1999.
[Kel12] David Kelly. Itô corrections in stochastic equations. PhD thesis, University of Warwick, 2012.
[Kre00] Dirk Kreimer. Shuffling quantum field theory. Lett. Math. Phys., 51(3):179191, 2000.
[LCL07] Terry J. Lyons, Michael Caruana, and Thierry Lévy. Differential equations driven by rough paths. Springer, 2007.
[LOT21] Pablo Linares, Felix Otto, and Marcus Tempelmayr. The structure group for quasi-linear equations via universal enveloping algebras. arXiv preprint arXiv2103.04187, 2021.
[Lyo98] Terry J. Lyons. Differential equations driven by rough signals. Rev. Mat. Iberoamericana, 14(2):215-310, 1998.
[Man11] Dominique Manchon. A short survey on pre-Lie algebras. In Noncommutative geometry and physics: renormalisation, motives, index theory, ESI Lect. Math. Phys., pages 89-102. Eur. Math. Soc., Zürich, 2011.
[Mil84] John W. Milnor. Remarks on infinite-dimensional Lie groups. In Relativity, groups and topology, II (Les Houches, 1983), pages 1007-1057. NorthHolland, Amsterdam, 1984.
[MM65] John W. Milnor and John C. Moore. On the structure of Hopf algebras. Ann. of Math. (2), 81:211-264, 1965.
[MP10] Dominique Manchon and Sylvie Paycha. Nested sums of symbols and renormalized multiple zeta values. Int. Math. Res. Not. IMRN, 24:4628-4697, 2010.
[MP18] Frédéric Menous and Frédéric Patras. Renormalization: a quasi-shuffle approach. In Computation and combinatorics in dynamics, stochastics and control, volume 13 of Abel Symp., pages 599-628. Springer, Cham, 2018.
[Pre16] Rosa Preiß. From Hopf algebras to rough paths and regularity structures. Master's thesis, Technische Universität Berlin, 2016. http://page.math.tu-berlin.de/~preiss/files/masters.pdf.
[Pre21] Rosa Preiß. Hopf algebras and non-associative algebras in the study of iterated-integral signatures and rough paths. PhD thesis manuscript, submitted to Technische Universität Berlin, 2021. http://page.math.tu-berlin.de/~preiss/files/submittedDissertation.pdf.
[PSS19] Max Pfeffer, Anna Seigal, and Bernd Sturmfels. Learning paths from signature tensors. SIAM J. Matrix Anal. Appl., 40(2):394-416, 2019.
[Rah21] Ludwig Rahm. Translations of rough paths in combinatorial hopf algebras. arxiv preprint arXiv:2111.02876, 2021.
[Reu93] Christophe Reutenauer. Free Lie Algebras. LMS monographs. Clarendon Press, 1993.
[TZ20] Nikolas Tapia and Lorenzo Zambotti. The geometry of the space of branched rough paths. Proc. Lond. Math. Soc. (3), 121(2):220-251, 2020.


[^0]:    ${ }^{1}$ (Warning.) We insist again that smooth rough paths are much richer objects than canonical lifts of smooth paths, unfortunately also called smooth rough paths in the earlier stages of the theory. To wit, the pure area rough path, familiar in rough path theory and seen later on in the text, is a perfect example of a smooth rough path.

[^1]:    ${ }^{2}$ In fact, a Hopf algebra on $T\left(\mathbb{R}^{d}\right)$ together with the deconcatenation coproduct, see e.g. Reu93, though this will not play a role in this section.

[^2]:    ${ }^{3}$ Even though $d=1$ weakly geometric rough paths are an outlier in the sense that they are fully determined by the underlying path, they can be trivially embedded into rough paths spaces of higher dimension and thus such a counterexample also applies there.

[^3]:    ${ }^{4}$ The backward path is given by $\overleftarrow{\mathbf{X}}_{s, t}=\mathbf{X}_{(T-s),(T-t)}$.

